Ranking Asymmetric Auctions with Several Bidders*

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March 2016

Abstract

Ranking the profitability of the first-price auction (FPA) and the second-price auction (SPA) is of fundamental importance to auction theory. However, the theoretical literature on bidder asymmetry has primarily focused on auctions with two bidders. Here, I consider auctions with several asymmetric bidders. As in the empirical literature, it is assumed that any bidder is either weak or strong. There is no unambiguous revenue ranking in this environment. Indeed, I show that the ranking may depend on both the size of the reserve price and the number of bidders. However, there always exists a range of reserve prices for which the FPA strictly dominates the SPA. Moreover, if the asymmetry is not too large, there exists seller own-use valuations for which the FPA with an optimal reserve price is strictly more profitable than the SPA with an optimal reserve price. The FPA may in fact be both more profitable and more efficient than the SPA when the reserve price is endogenous. These results are founded on the methodological insight that the combination of reserve prices and several bidders may allow the use of mechanism design arguments that are simpler than those required when just two bidders are present.

JEL Classification Numbers: D44, D82.

Keywords: Asymmetric Auctions, Reserve Prices, Revenue Ranking.

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*I would like to thank the Canada Research Chairs programme and the Social Sciences and Humanities Research Council of Canada for funding this research.

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1 Introduction

Starting with Vickrey's (1961) seminal work, a central preoccupation of auction theory has been to rank the profitability of different types of auctions. Vickrey (1961) himself was first to identify asymmetry among bidders as a critical factor. Although auction theory since then has evolved and expanded in many directions, the impact of bidder asymmetry is still less than perfectly understood.

In Vickrey's (1961) model, the first-price auction (FPA) and the second-price auction (SPA) are revenue equivalent when bidders are symmetric. However, by examining a specialized auction setting with two bidders, Vickrey also demonstrated that no arbitrary ranking can be obtained when bidders are asymmetric. In a well-known paper, Maskin and Riley (2000) developed a few general principles for when the FPA dominates the SPA, and vice versa. However, they also concentrate on settings with two bidders, one of which is ex ante perceived as "strong" and the other as "weak". Kirkegaard (2012a) generalized Maskin and Riley's (2000) insights using a mechanism design approach. Indeed, his results carry through with more weak bidders, yet the proof strategy breaks down when more strong bidders are present. The current paper is the first to systematically challenge the two-bidder assumption.[1]

The empirical literature on asymmetric auctions is larger and is growing more rapidly. This literature permits a larger number of bidders but generally lumps them into two groups. Campo, Perrigne, and Vuong (2003) divide bidders into solo bidders and joint bidders. In De Silva, Dunne, and Kosmopoulou (2003) bidders are either entrants or incumbents. The bidders in Flambard and Perrigne's (2006) study are located in one of two areas. Brendstrup and Paarsch (2006) consider an application with major and minor bidders. Likewise, in Marion (2007) and Krasnokutskaya and Seim (2011) bidders are classified as either large or small. Finally, Athey, Levin, and Seira (2011) put loggers and sawmills in separate groups. These examples demonstrate the empirical relevance of asymmetric auctions in a wide variety of settings. At the same time, they illustrate a growing interest in the topic among economists more

[1] Bidding behavior with an arbitrary number of bidders has been studied in detail by e.g. Lebrun (1999, 2006) and Kirkegaard (2009). However, this literature does not compare expected revenue across auctions. A number of papers rank the two auctions under more specialized conditions. Lebrun's (1996) and Cheng's (2006) results come from analytically studying environments with power distributions. Gavious and Minchuk (2014) use perturbation analysis to rank auctions with small asymmetries. Several papers use numerical methods to obtain a ranking; see e.g. Marshall et al (1994) and Li and Riley (2007). See Kirkegaard (2012a) for more references.

broadly. Given the sparsity of theoretical results, the empirical literature is forced to resort to numerical analysis when comparing expected revenue from the observed auction format to expected revenue from some counterfactual auction format. However, such analysis is not designed to uncover economic principles that yield deeper insights into what determines the relative profitability of different auctions. Thus, there remains a need for theory to help generate economic insights.

Consistent with practice in the aforementioned literature, the main simplifying assumption of this paper is that bidders can be divided into a strong and a weak group. As alluded to earlier, the analysis changes in qualitatively important ways as soon as there is more than one strong bidder. In particular, it is no longer necessarily true that both groups of bidders tender bids in the same range; there may exist a range of high bids which only strong bidders would consider submitting in equilibrium. This phenomenon is henceforth referred to as bid-separation.

The model used here is general enough to fit all of Maskin and Riley's (2000) examples as limiting cases. Therefore, an unambiguous revenue ranking does not exist in the model. Contrary to Maskin and Riley (2000) and Kirkegaard (2012a), the objective of the paper is not to isolate more refined conditions under which one auction is always better than the other. Instead, a primary contribution of the paper is to show that *all* environments consistent with the general and sparsely structured model share a common property; the FPA is strictly more profitable than the SPA for a range of reserve prices. This observation serves as a starting point for examining auctions with endogenous reserve prices. It is important to note that the optimal reserve price typically depends on the auction format. This has implications for efficiency since the magnitude of the reserve price determines how often a gain from trade is realized. A key result is that once the reserve price is endogenized, the FPA may be both more profitable and more efficient than the SPA.

The paper is founded on a new methodological insight that can be broken into two parts. First, as explained momentarily, elementary mechanism design arguments can be used to rank the two auctions if equilibrium in the FPA features "enough" bid-separation. However, it is endogenous whether bid-separation occurs, and if so to what extent. Nevertheless, recall that bid-separation never occurs in auctions with just one strong bidder. Hence, the addition of more strong bidders perhaps counterintuitively opens the door for simpler mechanism design arguments to be utilized. The second observation is that the equilibrium properties of the FPA – including the incidence of bid-separation – are influenced by the size of the reserve price. Stated differently, the reserve price provides a lever that can be used to manipulate equilibrium behavior. Thus, it is possible to "trigger" the right amount of bid-separation and invoke the simpler arguments.[2] Reserve prices are thus central to the approach.

To illustrate the method, begin by considering the extreme case where the reserve price is so high that weak bidders are de facto excluded. Only strong bidders compete. Thus, a form of bid-separation is taking place. The FPA and SPA are revenue equivalent since they allocate the good the same way. Next, lower the reserve price marginally to afford weak bidders with high valuations a small chance of winning. It is well-known that participating weak bidders bid more aggressively than strong bidders with comparable types. Nevertheless, by continuity, the allocation in the FPA is still "close to" efficient. Thus, weak bidders win slightly more often than is efficient. Standard mechanism design arguments then imply that the FPA dominates the SPA. This is in contrast to two-bidder auctions where Maskin and Riley (2000) argue that the weak bidder at times wins excessively often in a FPA from a design perspective. Kirkegaard (2012a) uses more refined mechanism design arguments to establish a revenue ranking, but those arguments break down if there is more than one strong bidder. His results also rely on stronger assumptions on type distributions than in the current paper. In summary, leveraging reserve prices allows the use of design arguments that are much simpler than in the case with a single strong bidder.

This revenue ranking relies on a relatively high reserve price. In fact, the SPA may outperform the FPA at small reserve prices. In other words, the ranking of the two auctions may flip when the reserve price changes. For a fixed reserve price, the ranking may likewise change as more bidders join the auction. An example exhibiting these reversal properties is provided. To my knowledge, this is the first example that demonstrates these properties. The reason that this is the first such example is that the existing literature has more or less deliberately avoided comparative statics of this nature. First, focus has been on auctions with two bidders. Second, the additional assumptions imposed in e.g. Maskin and Riley (2000) and Kirkegaard (2012a) are so strong that the revenue ranking then turns out to be the same regardless of the reserve prices. In the setting in the current paper, the FPA dominates the SPA for a

[2] Kirkegaard (2012b) provides two examples in which the FPA can be shown to dominate the SPA with an arbitrary number of weak and strong bidders. These examples require that the asymmetry between the two groups is exogenously large enough to trigger bid-separation. The point of the current paper is that a reserve price in effect allows one to endogenize the size of the asymmetry.

wider and wider range of reserve prices as more bidders participate in the auction.

However, the optimal reserve price is endogenous. It depends on the auction format, the number of bidders, and the seller's own-use valuation. Unless the asymmetry is very large, there exists own-use valuations for which the FPA with an optimal reserve price is strictly more profitable than the SPA with an optimal reserve price when there are sufficiently many bidders.[3] Indeed, there are own-use valuations for which the same strict profit ranking obtains if the asymmetry is small enough, regardless of the number of bidders. Both results can be interpreted as "impossibility result" since they imply that the SPA cannot be weakly better than the FPA for all own-use valuations and all combinations of bidders. Hence, any finding that the SPA is superior to the FPA in some specific setting is fragile to changes along these dimensions. More succinctly, the SPA cannot be "robustly" superior to the FPA in this model. In contrast, there are settings where the FPA weakly dominates the SPA for all own-use valuations and any combination of bidders. Generally, however, the profit ranking may change as the seller's own-use valuation changes. Empirical implications of these result are discussed further in Section 7.

Ignoring reserve prices, the received wisdom is that the SPA is more efficient than the FPA when bidders are risk neutral but asymmetric; see e.g. Krishna (2002). Specifically, total surplus is higher in the SPA than in the FPA when the reserve price is the same in both auctions. However, the SPA does not Pareto dominate the FPA since weak bidders prefer the latter auction format. Now, the optimal reserve price is lower in the FPA than in the SPA when the seller's own-use valuation is high. Thus, the good is sold more often in the FPA. On the other hand, the allocation contingent on a sale need not be efficient in the FPA. A priori then, it is not obvious which auction is more efficient once the reserve price is endogenized. I give an example in which the FPA with an optimal reserve price is more efficient than the SPA with an optimal reserve price. In fact, in the example the FPA is Pareto superior to the SPA. That is, both the seller and the buyers prefer the former. Again, this is the first such example that I am aware of.[4] Thus, it must be recognized that it is generally an empirical question which auction is more efficient once the reserve price is endogenized.

[3] If the asymmetry is large, the optimal reserve price in either auction may be so high that it excludes the weak bidders. Then, the seller is indifferent between the two auctions.

[4] It is known that optimal reserve prices may differ across auction formats if bidders are symmetric but risk averse. The implications for efficiency are examined in Hu, Matthews, and Zou (2010).

2 Model

Two groups of risk neutral bidders participate in a FPA or SPA. Bidders in the strong group independently draw a valuation from the twice continuously differentiable distribution function $F_s(v)$, with support $[\underline{v}_s, \overline{v}_s]$. The density, $f_s(v)$, is assumed to be strictly positive for all $v \in (\underline{v}_s, \overline{v}_s]$. Note that mass points are ruled out. There are a total of $m_s \geq 2$ strong bidders. There are also $m_w \geq 1$ weak bidders. These bidders independently draw a valuation from another twice continuously differentiable distribution function $F_w(v)$, $v \in [\underline{v}_w, \overline{v}_w]$. Again, it is assumed that the density $f_w(v)$ is strictly positive for all $v \in (\underline{v}_w, \overline{v}_w]$. Assume that $\overline{v}_s > \overline{v}_w > \underline{v}_s \geq \underline{v}_w$. Thus, the supports overlap. Finally, it is assumed that F_s dominates F_w in terms of the reverse hazard rate, $F_w \leq_{rh} F_s$, or

$$\frac{f_s(v)}{F_s(v)} \geq \frac{f_w(v)}{F_w(v)} \text{ for all } v \in (\underline{v}_s, \overline{v}_w]. \tag{1}$$

In other words, $\frac{F_s(v)}{F_w(v)}$ is non-decreasing on $(\underline{v}_s, \overline{v}_w]$. Hence, a strict version of first order stochastic dominance applies since $F_s(v) < F_w(v)$ for all $v \in (\underline{v}_s, \overline{v}_w]$. A generic member of the strong (weak) group is for simplicity referred to as bidder s (w). The number and composition of bidders, i.e. m_s and m_w, are exogenous.

In much of the paper a non-trivial – and possibly endogenous – reserve price, r, is imposed. Specifically, the reserve price r is non-trivial when $r > \underline{v}_s$, in which case the good will remain unsold with positive probability. Since there are at least two strong bidders, the reserve price has no effect on equilibrium if it is no greater than $\underline{v}_s$, or $r \leq \underline{v}_s$. Thus, assume without loss of generality that $r \geq \underline{v}_s$, where the special case $r = \underline{v}_s$ is equivalent to the absence of a reserve price. Any bidder with type strictly below r does not submit a bid in the FPA or SPA.

3 Equilibrium and comparative statics of the FPA

Lebrun (2006) characterizes equilibrium in the FPA under more general assumptions than those stated above. He proves equilibrium is unique whenever $r > \underline{v}_s$. For any $r \in [\underline{v}_s, \overline{v}_w]$, bidder i with type $v \geq r$ submits a bid in the interval $[r, \overline{b}_i]$, $i = s, w$. Naturally, $\overline{b}_s$ and $\overline{b}_w$ are endogenously determined, with $\overline{b}_s \geq \overline{b}_w$. Thus, all bidders submit bids in the same range if and only if $\overline{b}_s = \overline{b}_w$. This is the case in Maskin and Riley (2000) and Kirkegaard (2012a), where $m_s = m_w = 1$. If $\bar{b}_s > \bar{b}_w$ in such a setting, the lone strong bidder who is supposed to bid $\bar{b}_s$ would profit by slightly lowering his bid since it would not affect his chances of winning. That argument breaks down as soon as $m_s \geq 2$, as in the current paper. In fact, the central arguments of the paper rely on the possibility that $\bar{b}_s > \bar{b}_w$. The term bid-separation is henceforth used to refer to any equilibrium in which $\bar{b}_s > \bar{b}_w$.

In equilibrium, there exists a unique threshold type, $\widehat{v}$, such that bidder s with type $\widehat{v}$ bids $\bar{b}_w$ in the FPA. Higher types separate away from weak bidders by bidding above $\bar{b}_w$. In contrast, types below $\widehat{v}$ engage with weak bidders and may thus potentially lose to a weak bidder. Note that bid-separation takes place if and only if $\widehat{v} < \overline{v}_s$. While bid-separation is implicit in Lebrun's (2006) equilibrium characterization, Hubbard and Kirkegaard (2015) examine this equilibrium feature more closely. They also assume bidders belong to one of two groups, but they do not assume reverse hazard rate dominance. Likewise, although they present several comparative statics results, they do not consider changes in the reserve price. However, they prove that

$$\widehat{v} = \min \left\{ \overline{v}_s, \frac{m_s}{m_s - 1} \overline{v}_w - \frac{1}{m_s - 1} \bar{b}_w \right\}. \tag{2}$$

Now, since $\bar{b}_w$ is bounded between r and $\overline{v}_w$, the above relationship proves formally that $\widehat{v}$ converges to $\overline{v}_w$ as r converges to $\overline{v}_w$. In other words, bid-separation must occur when the reserve price is high enough and $m_s \geq 2$. Note also that $\widehat{v} > \overline{v}_w$ for any $r < \overline{v}_w$. Thus, a weak bidder with type $\overline{v}_w$ outbids strong bidders with higher types. In other words, he wins more often than is efficient.

Given some endogenous $(\widehat{v}, \bar{b}_w)$, the challenge is to describe equilibrium behavior at bids between r and $\bar{b}_w$, or accounting for how weak and strong bidders with types below $\overline{v}_w$ and $\widehat{v}$ bid, respectively. At bids above $\bar{b}_w$, the auction is essentially a symmetric auction since only strong bidders have types that are active there. Given $(\widehat{v}, \bar{b}_w)$, it is thus trivial to describe the bidding behavior of strong bidders with types above $\widehat{v}$; see Hubbard and Kirkegaard (2015).

Lebrun (2006) and Hubbard and Kirkegaard (2015) characterize equilibrium of the FPA by describing inverse bidding strategies. However, from a mechanism design perspective it is often more fruitful to characterize the equilibrium allocation instead. Thus, as in Kirkegaard (2012a), the problem is reformulated. Consider a weak bidder with type $v \geq r$. Let $b_w(v)$ denote his equilibrium bid. In equilibrium, this bid equals the bid submitted by a strong bidder with some type $k(v)$. Hence, the weak bidder wins if and only if all the other weak bidders have type below v and all the strong bidders have type below $k(v)$. For bids below $\bar{b}_w$, equilibrium can thus be characterized by describing the pair of (b_w, k) functions instead of the pair of inverse bidding functions. In either case, the endogenous functions solve a system of differential equations with appropriate boundary conditions and initial conditions. In the formulation used here, the boundary conditions are that $k(\bar{v}_w) = \widehat{v}$ and $b_w(\bar{v}_w) = \bar{b}_w$. The initial conditions are described later. The relevant system of differential equations is described in the beginning of Appendix A.

Equilibrium depends on the parameters (r, m_s, m_w). Thus, I generally write the endogenous functions as $k(v|r, m_s, m_w)$ and $b_w(v|r, m_s, m_w)$, respectively, but make use of the shorter form $k(v)$ and $b_w(v)$ whenever no confusion arises as a result.

Note that a weak bidder with type v bids more aggressively than a strong bidder with type v if and only if $k(v) > v$. Recall that $k(\bar{v}_w) = \widehat{v} > \bar{v}_w$ when $r < \bar{v}_w$. Indeed, it is a standard result that reverse hazard rate dominance implies $k(v) > v$ globally; see e.g. Lebrun (1999) and Maskin and Riley (2000) for proofs of this result in various settings. The following lemma proves that the property holds in the present setting.

Lemma 1 *Assume $r \in [\underline{v}_s, \bar{v}_w)$. Then, $k(v) > v$ for all $v \in (r, \bar{v}_w]$.*

Proof. See Appendix A. ■

Several results rely on how the allocation changes with the reserve price. The easiest way to see that the allocation must change is to note that the "initial conditions" to the system of differential equations change. In particular, combining Lemma 1 and Lebrun's (2006) analysis implies that $b_w(r) = r$ and $k(r) = r$, as explained in the proof of Proposition 1. The first comparative statics result is a monotonicity result. Specifically, $k(v)$ is decreasing in r as illustrated in the left panel of Figure 1.

Proposition 1 *Assume $m_s \geq 2$, $m_w \geq 1$. If $\bar{v}_w > r' > r \geq \underline{v}_s$ then*

$$k(v|r', m_s, m_w) < k(v|r, m_s, m_w) \text{ for all } v \in [r', \bar{v}_w).$$

Proof. See Appendix A. ■

Consider a weak bidder with some type $v \in [r', \bar{v}_w)$. When the reserve price increases from r to r', this bidder becomes less likely (Proposition 1) to outbid the strong bidders and win the FPA. However, it is still the case that he wins more often than is efficient (Lemma 1). Increasing the number of bidders has a similar effect.

Proposition 2 *Assume* $m_s' > m_s \geq 2$, $m_w' > m_w \geq 1$, $r > \underline{v}_s$, *and* $\overline{v}_s > \widehat{v} = k(\overline{v}_w | r, m_s, m_w)$. *Then,*

$$k(v|r, m_s', m_w) < k(v|r, m_s, m_w) \text{ and } k(v|r, m, m_w') < k(v|r, m_s, m_w) \text{ for all } v \in (r, \overline{v}_w).$$

Proof. See Appendix A. ■

The right panel of Figure 1 illustrates Proposition 2. This can also be thought of as a monotonicity result. In particular, the auction becomes closer and closer to efficient the more bidders are participating in the auction. Thus, this result complements Swinkels' (1999, 2001) finding that the first-price auction is asymptotically efficient. In other words, $k(v|r, m_s, m_w)$ converges to v as the number of bidders goes to infinity. The implication that $k(v|r, m_s, m_w)$ is not bounded away from v in the limit is useful. For completeness, the next result states and proves this fact.

Proposition 3 *Assume* $m_s \geq 2$, $m_w \geq 1$. *Then* $k(v|r, m_s, m_w) \to v$ *for all* $v \in (r, \overline{v}_w]$ *as* $m_s \to \infty$ *or* $m_w \to \infty$.

Proof. See Appendix A. ■

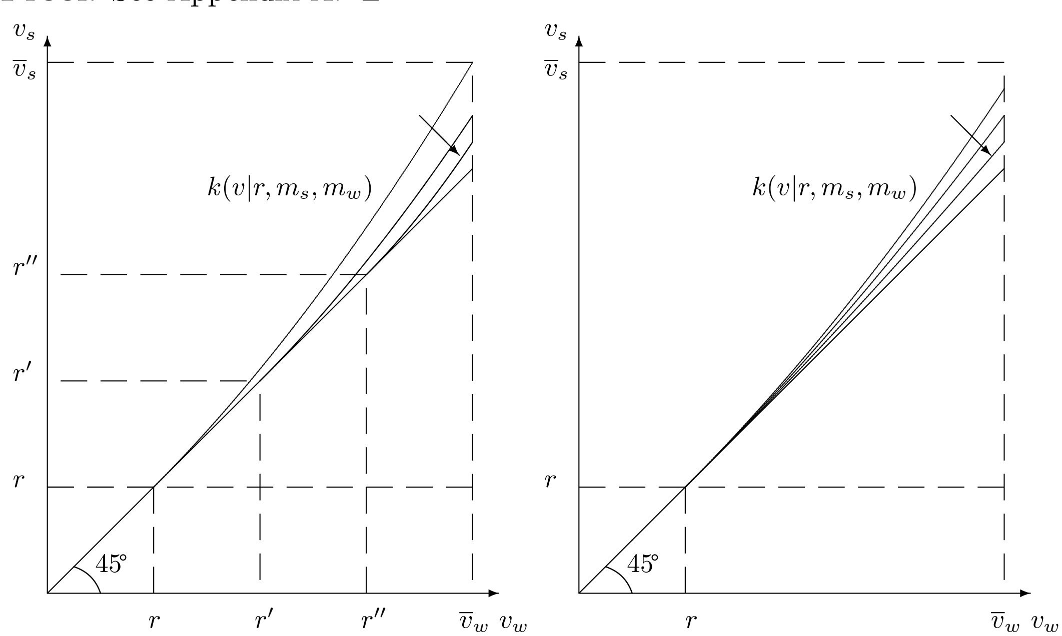

Figure 1: (a) The left panel depicts how $k(v|r, m_s, m_w)$ changes with r, given (m_s, m_w); (b) The right panel shows how $k(v|r, m_s, m_w)$ changes with (m_s, m_w).

4 Ranking auctions with large reserve prices

Myerson (1981) shows that expected revenue in any auction equals the expected value of the winner's virtual valuation.[5] Bidder i's virtual valuation is

$$J_i(v) = v - \frac{1 - F_i(v)}{f_i(v)}.$$

The comparative statics in the previous section are useful because they reveal how the allocation in the FPA depends on the reserve price and the composition of bidders. Let $ER^{FPA}(r, m_s, m_w)$ denote the expected revenue in the FPA given (r, m_s, m_w). However, even under the assumption that the seller is risk neutral, he may care about more than expected revenue. Let z denote the seller's own-use valuation. Then, his expected payoff in the FPA is

$$\Pi^{FPA}(z, r, m_s, m_w) = zF_s(r)^{m_s} F_w(r)^{m_w} + ER^{FPA}(r, m_s, m_w).$$

The literature often assumes implicitly or explicitly that $z = 0$. Of course, this is an innocent normalization if the reserve price is exogenous. However, when the reserve price is endogenous, its optimal value typically depends on z. Optimal reserve prices are examined Section 7. For now, the reserve price is thought of as exogenous.

Although there are multiple equilibria in the SPA, I focus on the equilibrium in which bidders use the weakly dominant strategy of bidding truthfully. When it is sold, the good is thus allocated to the bidder with the highest type. Let $ER^{SPA}(r, m_s, m_w)$ and $\Pi^{SPA}(z, r, m_s, m_w)$ denote the expected revenue and the expected payoff to the seller in the SPA, respectively. Recall that the weak bidder with the highest valuation wins more often in the FPA than in the SPA, since $k(v) > v$ for $v > r$. Hence, as noted by Kirkegaard (2012a), for a fixed reserve price, $r \in [\underline{v}_s, \overline{v}_w]$, the revenue difference between the two auctions is

$$\begin{aligned} \Delta(r, m_s, m_w) &= ER^{FPA}(r, m_s, m_w) - ER^{SPA}(r, m_s, m_w) \\ &= \Pi^{FPA}(z, r, m_s, m_w) - \Pi^{SPA}(z, r, m_s, m_w) \\ &= \int_r^{\overline{v}_w} \left(\int_v^{k(v|r,m_s,m_w)} (J_w(v) - J_s(x))\, dF_s(x)^{m_s} \right) dF_w(v)^{m_w}. \quad (3) \end{aligned}$$

[5] This statement is true whenever any bidder earns zero payoff when he has the lowest possible type in his type support. That property holds here due to the assumptions that $m_s \geq 2$ and $\underline{v}_s \geq \underline{v}_w$.

Intuitively, the inner integral in (3) captures the fact that when the most competitive of the weak bidders wins in the FPA but not in the SPA it is because the most competitive bidder in the strong group has a type above v but below $k(v)$. If the reserve price is so high that weak bidders are excluded, or $r \geq \overline{v}_w$, then the two auctions allocate the object in the same way. In this case the SPA and FPA are revenue equivalent, or $\Delta(r, m_s, m_w) = 0$.

Expected revenue in the FPA is strictly higher than expected revenue in the SPA if the parameters (r, m_s, m_w) belong to the set

$$\mathcal{P} = \{(r, m_s, m_w) | J_w(v) - J_s(x) > 0 \text{ for all } x \in [v, k(v|r, m_s, m_w)] \text{ and all } v \in (r, \overline{v}_w]\},$$

in which case each term in the inner integral in (3) is strictly positive. In this case, when the allocation in the FPA differs from the allocation in the SPA it is because the item has been awarded to a bidder with a strictly higher virtual valuation.

However, Maskin and Riley (2000) point out that

$$J_s(\overline{v}_s) > J_w(\overline{v}_w) > J_s(\overline{v}_w). \tag{4}$$

Hence, from a revenue perspective it is desirable that the weak bidder with type $\overline{v}_w$ wins more often than is efficient. However, he wins too often if he outbids strong bidders with types close to $\overline{v}_s$. Note that this must necessarily occur if there is no bid-separation, as is the case in any two-bidder model or more generally if $m_s = 1$. Stated differently, there is no $(r, m_s, m_w) \in \mathcal{P}$ for which $m_s = 1$. Thus, Maskin and Riley (2000) conclude that "mechanism design considerations do not settle the matter of which auction generates more revenue." The innovation in Kirkegaard (2012a) is based on the observation that what is important is not whether the winner's virtual valuation is no lower in the FPA than in the SPA with probability one, but rather whether this is the case in expectation. Hence, he identifies conditions under which the inner integral in (3) is positive. However, Kirkegaard (2012a) explicitly makes the point that his method may fail if there is more than one strong bidder.

As discussed previously, however, the possibility of bid-separation is a distinguishing feature of auctions with more than one strong bidder. Bid-separation limits how often any weak bidders wins. Even if his type is $\overline{v}_w$, he wins only if all strong bidders have types below $\widehat{v}$. Since $\widehat{v}$ depends on the reserve price, the latter can be used as a lever to determine how often weak bidders win. As a result, the winner's virtual valuation in the FPA will be shown to be no smaller than in the SPA for at least some reserve prices. Formally, the idea is to show that there are $(r, m_s, m_w) \in \mathcal{P}$ with $m_s \geq 2$. This is precisely the simple proof strategy that Maskin and Riley (2000) note cannot work in two-bidder auctions. Thus, contrary to what common intuition may suggest, auctions with several bidders may be methodologically and conceptually simpler to handle than auctions with just two bidders.

Recall that $\widehat{v} = k(\overline{v}_w)$ converges to $\overline{v}_w$ from above as r converges to $\overline{v}_w$ from below. In other words, for any $v \in (r, \overline{v}_w]$, $k(v|r, m_s, m_w)$ can be made arbitrarily close to v by gradually increasing r. At the same time, (4) implies that $J_w(v) > J_s(x)$ when v and x are close to $\overline{v}_w$. Thus, as r increases towards $\overline{v}_w$, $(r, m_s, m_w) \in \mathcal{P}$. Hence, the FPA dominates the SPA for sufficiently large r.

Proposition 4 *Given $m_s \geq 2$ and $m_w \geq 1$, there exists an $\widehat{r} \in [\underline{v}_s, \overline{v}_w)$ such that $\Delta(r, m_s, m_w) > 0$ for all $r \in [\widehat{r}, \overline{v}_w)$.*

Proposition 4 is a "local" result that requires minimal assumptions; it has been assumed only that $\overline{v}_s > \overline{v}_w$ and that reverse hazard rate dominance applies. Indeed, it would be sufficient to assume that reverse hazard rate dominance applies "locally" around $\overline{v}_w$, or, by continuity, that $\frac{f_s(\overline{v}_w)}{F_s(\overline{v}_w)} > f_w(\overline{v}_w)$. Note that not even first order stochastic dominance is required to hold. An important implication is that under these minimal assumptions, there is no auction environment where the SPA dominates the FPA for all reserve prices. Thus, there is no hope of finding more refined conditions where the SPA dominates the FPA regardless of the reserve price. Note that Maskin and Riley's (2000) example in which the SPA is more profitable than the FPA assumes $\overline{v}_w = \overline{v}_s$; see Section 6.

Appendix C extends Proposition 4 to the case with a single strong bidder, $m_s = 1$. Since bid-separation never arises in that case, arguments that are closer in spirit to Kirkegaard (2012a) must be used. Appendix C also describes conditions that are weaker than $(r, m_s, m_w) \in \mathcal{P}$, but which are still sufficient for $\Delta(r, m_s, m_w) > 0$. Finally, note also that Proposition 4 must hold if there are more than two groups of bidders, when each group has a distinct maximum type. After all, the analysis applies when the reserve price is so high that it effectively excludes all but two groups.

5 Comparative statics

Two types of comparative statics are pursued here. I first analyze the effects of changing the composition of bidders. Second, I consider the consequences of increasing the degree of asymmetry between the two groups of bidders. The common conclusion is that the FPA dominates the SPA for more reserve prices as more bidders participate in the auction or as the asymmetry becomes more pronounced.

5.1 Varying the number of bidders

Propositions 1 and 2 imply that (r, m_s, m_w) is more likely to belong to $\mathcal{P}$ the higher r, m_s, or m_w are. The reason is that $k(v)$ decreases, meaning that the strict inequality in the definition of $\mathcal{P}$ must hold for fewer values of x (and fewer values of v if r increases too). Hence, if the sufficient conditions for ranking the FPA above the SPA are satisfied for some (r, m_s, m_w) triplet, then they are also satisfied when r increases or when the number of bidders increases.

Proposition 5 *Assume $\overline{v}_w > r' \geq r > \underline{v}_s$, $m_s' \geq m_s \geq 2$, and $m_w' \geq m_w \geq 1$. Then, $(r', m_s', m_w') \in \mathcal{P}$ if $(r, m_s, m_w) \in \mathcal{P}$.*

Proposition 5 implies that as the number of bidders increases, the set of reserve prices for which the FPA can be proven to be preferable to the SPA weakly expands. A stronger version can be obtained under the additional assumption that F_s strictly dominates F_w in terms of the hazard rate, $F_w <_{hr} F_s$, or

$$\frac{f_s(v)}{1 - F_s(v)} < \frac{f_w(v)}{1 - F_w(v)} \text{ for all } v \in (\underline{v}_s, \overline{v}_w).$$

Note that $J_w(v) > J_s(v)$ for all $v \in (\underline{v}_s, \overline{v}_w]$. Thus, it follows that $(r, m_s, m_w) \in \mathcal{P}$ if $k(v)$ is sufficiently close to v for all $v \in (r, \overline{v}_w]$. Invoking Propositions 2 and 3 then imply that for any $r \in [\underline{v}_s, \overline{v}_w)$, the FPA is strictly more profitable than the SPA when sufficiently many bidders are participating in the auction.

Proposition 6 *Assume $F_w <_{hr} F_s$. Then, for any $r \in [\underline{v}_s, \overline{v}_w)$, $(r, m_s, m_w) \in \mathcal{P}$ when m_s and/or m_w are sufficiently large.*

5.2 Growing asymmetry: Stretching F_s

There are several ways of formalizing the idea that the degree of asymmetry between the two groups of bidders changes. The modelling choice made here relies on the observation that $\overline{v}_s \neq \overline{v}_w$ is instrumental to the proof strategy. Holding $F_w(v)$ and $\overline{v}_w$ fixed, the idea is thus to describe changes in the strong group's distribution that allows for increases in $\overline{v}_s$ in such a way that the difference between $\overline{v}_s$ and $\overline{v}_w$ can be used as a meaningful measure of the degree of asymmetry.

Consider some twice continuously differentiable function, $G(v)$, defined for all $v \geq \underline{v}_s$. Assume that $G(\underline{v}_s) = 0$ and that the derivative, $g(v)$, is strictly positive for any $v > \underline{v}_s$. For any $\overline{v}_s \geq \overline{v}_w$, let

$$F_s(v|\overline{v}_s) = \frac{G(v)}{G(\overline{v}_s)}, \text{ for all } v \in [\underline{v}_s, \overline{v}_s] \tag{5}$$

and assume that

$$\frac{f_s(v|\overline{v}_s)}{F_s(v|\overline{v}_s)} = \frac{g(v)}{G(v)} \geq \frac{f_w(v)}{F_w(v)}, \text{ for all } v \in (\underline{v}_s, \overline{v}_w]. \tag{6}$$

Hence, F_s dominates F_w in terms of the reverse hazard rate for any $\overline{v}_s > \overline{v}_w$. Note that the two distributions may coincide in the limit where $\overline{v}_s \to \overline{v}_w$. Moreover, this formulation is without loss of generality. That is, for any fixed F_w and $\overline{v}_s$, any F_s that satisfies the assumptions in Section 2 can be written as (5) and must satisfy (6).

Adapting Maskin and Riley's (2000) terminology, increases in $\overline{v}_s$ amounts to "stretching" the distribution. Note that the strong groups' virtual valuation,

$$J_s(v|\overline{v}_s) = v - \frac{1 - F_s(v|\overline{v}_s)}{f_s(v|\overline{v}_s)} = v - \frac{G(\overline{v}_s) - G(v)}{g(v)}$$

is strictly decreasing in $\overline{v}_s$. When needed, let $\mathcal{P}(\overline{v}_s)$ denote the set $\mathcal{P}$ for some fixed $\overline{v}_s$. Compare two values of $\overline{v}_s$, $\overline{v}_s'$ and $\overline{v}_s''$, with $\overline{v}_s'' > \overline{v}_s'$. Assume that $(r, m_s, m_w) \in \mathcal{P}(\overline{v}_s')$. Thus, bid-separation is occurring when $\overline{v}_s = \overline{v}_s'$. Stretching F_s entails adding more high types to the strong bidders' type space. It is not surprising that these new types will also separate away from weak bidders by bidding above $\overline{b}_w$. In fact, bidding behavior for existing types do not change.[6] That is, $k(v|r, m_s, m_w)$ is unchanged.

[6] Technically, the reason is that the reverse hazard rate is unchanged when F_s is stretched. The system of differential equations in the common bid range is then unchanged as well.

Then, since $J_s(v|\overline{v}_s'') < J_s(v|\overline{v}_s')$, it now follows that $(r, m_s, m_w) \in \mathcal{P}(\overline{v}_s'')$.

Proposition 7 *If* $(r, m_s, m_w) \in \mathcal{P}(\overline{v}_s')$ *then* $(r, m_s, m_w) \in \mathcal{P}(\overline{v}_s'')$ *for all* $\overline{v}_s'' > \overline{v}_s'$.

Proposition 7 implies that as the degree of asymmetry increases, the FPA can be proven to be superior to the SPA for more and more reserve prices. For a converse, consider values of $\overline{v}_s$ for which bid-separation arises.[7] Then, $k(v|r, m_s, m_w)$ is unchanged as $\overline{v}_s$ increases further, while $J_s(v|\overline{v}_s)$ strictly decreases. Thus, for any $r \in (\underline{v}_s, \overline{v}_w)$, the FPA dominates the SPA when the asymmetry is large enough.[8] Assuming away any reserve price, Kirkegaard (2012b) made a similar observation in more specialized settings.

6 Reversals of the revenue ranking

Compared to much other work on ranking asymmetric auctions, the structure imposed here is rather sparse. The model has been endowed only with the following properties:

(i) Different maximal types; $\overline{v}_w < \overline{v}_s$.

(ii) Reverse hazard rate dominance; $\frac{f_s(v)}{F_s(v)} \geq \frac{f_w(v)}{F_w(v)}$ for all $v \in (\underline{v}_s, \overline{v}_w]$.

Proposition 6 additionally assumes:

(iii) Strict hazard rate dominance; $\frac{f_s(v)}{1-F_s(v)} < \frac{f_w(v)}{1-F_w(v)}$ for all $v \in (\underline{v}_s, \overline{v}_w)$.

In Kirkegaard (2012a) and two of Maskin and Riley's (2000) examples, (i) and (ii) are imposed together with a stronger version of (iii). As explained in Kirkegaard (2012a), their assumptions are strong enough to guarantee that the FPA dominates the SPA for all reserve prices in the two-bidder case or when $m_s = 1$. Maskin and Riley (2000) present a third example in which the SPA dominates the FPA. They assume there are two bidders and no reserve price. However, their logic extends to any reserve price and any number of bidders. In that example, the inequalities in (i) and (iii) are replaced by equalities. Since this is a limiting case of the current

[7] It is implied by (2) that bid-separation must occur if $\overline{v}_s$ is sufficiently high, regardless of r.

[8] On the other hand, reserve prices above $\overline{v}_w$ become more profitable too. This is one of the reasons that endogenizing the reserve price is more challenging. See Section 7.

model, it follows that a general revenue ranking does not exist in the present set-up. It is worth reiterating that Kirkegaard's (2012a) model and Maskin and Riley's (2000) various examples all yield the very strong result that a reserve price cannot influence the revenue ranking. In comparison, the following example illustrates some key properties of the model in this paper.

EXAMPLE 1 (RANKING REVERSALS): Let $F_s(v)$ be obtained by truncating F_w on the left, such that

$$F_s(v) = \frac{F_w(v) - F_w(\underline{v}_s)}{1 - F_w(\underline{v}_s)},\ v \in [\underline{v}_s, \overline{v}_w]$$

for some truncation point $\underline{v}_s$, $\underline{v}_s \in (\underline{v}_w, \overline{v}_w)$. It is easy to see that $\frac{F_s(v)}{F_w(v)}$ is strictly increasing on $v \in (\underline{v}_s, \overline{v}_w]$. That is, reverse hazard rate dominance applies. However, contrary to the main model, $\overline{v}_w = \overline{v}_s$. It also holds that $J_w(v) = J_s(v)$ for all $v \in [\underline{v}_s, \overline{v}_w]$. Finally, assume that $J_w(v)$ and $J_s(v)$ are strictly increasing in v. Then, the efficient SPA allocates the good optimally whenever it is sold. In the FPA, bid-separation does not arise in equilibrium since $\overline{v}_w = \overline{v}_s$. However, due to reverse hazard rate dominance it must hold that $k(v) > v$ for all $v \in (r, \overline{v}_w)$, for any $r \in [\underline{v}_s, \overline{v}_w)$. Thus, weak bidders wins more often than is efficient. Hence, the SPA strictly dominates the FPA. Maskin and Riley's (2000) example that demonstrates the SPA may be more profitable than the FPA is based on the same logic.

Now perturb the model. Specifically, "stretch" F_s from the support $[\underline{v}_s, \overline{v}_w]$ to the support $[\underline{v}_s, \overline{v}_w + \varepsilon]$, where $\varepsilon > 0$ is small. The new, perturbed, distribution H_s satisfies $H_s(v) = \lambda F_s(v)$ for all $v \in [\underline{v}_s, \overline{v}_w]$ for some non-negative λ that is strictly smaller than one but very close to one. As in Section 5.2, the reverse hazard rate is unaffected on $[\underline{v}_s, \overline{v}_w]$ and so it still holds that H_s strictly reverse hazard rate dominates F_w. However, it is now the case that $\overline{v}_w < \overline{v}_w + \varepsilon = \overline{v}_s$ and $J_w(v) > J_s(v)$ for all $v \in [\underline{v}_s, \overline{v}_w]$. Hence, the perturbed model satisfies all the assumptions required for the previous analysis. Fix some $r \in [\underline{v}_s, \overline{v}_w)$ and some $m_s \geq 2$, $m_w \geq 1$. By continuity (see Lebrun (2002)), if ε is close enough to zero and λ close enough to one then it must still hold that the SPA dominates the FPA. Now increase the reserve price. As the reserve price approaches $\overline{v}_w$, Proposition 4 comes into effect.

In conclusion, the SPA dominates the FPA when the reserve price is low enough, whereas the FPA dominates the SPA when the reserve price is high enough. Similarly, holding r fixed, the SPA dominates the FPA with the original set of bidders. However, by Proposition 6, the FPA will eventually come to dominate the SPA as the number of bidders increases. I am aware of no other work that has demonstrated either of these ranking reversal properties before. ▲

Example 1 demonstrates that the current model has richer features than existing models as the revenue ranking may depend on both the size of the reserve price and the composition of bidders. On the other hand, the model yields the robust prediction that the FPA dominates the SPA for high reserve prices or many bidders.

7 Optimal reserve prices

7.1 Comparing reserve prices across auctions

Recall that when $(r, m_s, m_w) \in \mathcal{P}$, it necessarily holds that $J_w(v) - J_s(k(v)) > 0$ for all $v \in (r, \overline{v}_w]$. Then, by continuity, $J_w(v) - J_s(x) > 0$ even if x is slightly above $k(v)$. Thus, expected revenue in the FPA would be even higher if $k(v)$ was marginally higher. However, a drawback of increasing r in the FPA is that $k(v)$ declines further, by Proposition 1. This disadvantage is absent in the SPA. For this reason, there is less of an incentive to marginally increase r in the FPA than in the SPA.

Proposition 8 *Fix $r' \in (\underline{v}_s, \overline{v}_w)$ and assume $(r', m_s, m_w) \in \mathcal{P}$. Then,*

$$\frac{\partial \Delta(r, m_s, m_w)}{\partial r} < 0 \text{ for all } r \in [r', \overline{v}_w).$$

Proof. See Appendix A. ■

Let $r^{SPA}(z, m_s, m_w)$ denote the optimal reserve price in the SPA. If the optimal reserve price is not unique, then $r^{SPA}(z, m_s, m_w)$ denotes the *smallest* optimal reserve price. Let $r^{FPA}(z, m_s, m_w)$ denote *any* optimal reserve price in the FPA. With some abuse of notation, I write $r^{SPA}(z)$ and $r^{FPA}(z)$ whenever the number of bidders is understood to be fixed.

Assume that $r^{SPA}(z) \in (\underline{v}_s, \overline{v}_w)$ and that $(r^{SPA}(z), m_s, m_w) \in \mathcal{P}$. Proposition 8 now makes it possible to determine in which auction the optimal reserve price is larger and which auction is more profitable.

Corollary 1 *Assume $r^{SPA}(z) \in (\underline{v}_s, \overline{v}_w)$ and $(r^{SPA}(z), m_s, m_w) \in \mathcal{P}$. Then,*

$$r^{FPA}(z) < r^{SPA}(z)$$

and

$$\Pi^{FPA}(z, r^{FPA}(z), m_s, m_w) > \Pi^{SPA}(z, r^{SPA}(z), m_s, m_w).$$

Proof. See Appendix A. ■

Since $r^{SPA}(z) > r^{FPA}(z)$, the FPA is more likely to realize a gain from trade. The two auctions are compared on efficiency grounds in Section 9.

Note that Proposition 8 implies that $\Delta(r, m_s, m_w)$ is maximized to the right of any r for which $(r, m_s, m_w) \in \mathcal{P}$. That is, the difference between the expected revenue of the FPA and the SPA is maximized at a "small" reserve price.

On the other hand, in the first part of Example 1 – where $\overline{v}_w = \overline{v}_s$ and $J_w(v) = J_s(v)$ – it never holds that $(r^{SPA}(z), m_s, m_w) \in \mathcal{P}$. Indeed, in that example, there is an incentive to lower $k(v)$ in the FPA by increasing r beyond $r^{SPA}(z)$. Hence, it is also possible that $r^{SPA}(z) \leq r^{FPA}(z)$.

7.2 Ranking auctions with many bidders

Recall that $J_w(\overline{v}_w) = \overline{v}_w > 0$. Assume in this subsection that $J_s(v) \geq 0$ for all $v \in [\overline{v}_w, \overline{v}_s]$. One interpretation of the assumption is that the asymmetry between bidders is not too large. Assume moreover that z is small enough that $0 \leq z \leq J_s(v)$ for all $v \in [\overline{v}_w, \overline{v}_s]$. These assumptions are easily verified to imply that $\Pi^{SPA}(z, r, m_s, m_w)$ and $\Pi^{FPA}(z, r, m_s, m_w)$ are non-increasing in r for $r \geq \overline{v}_w$ (recall that the auctions are revenue equivalent for $r \geq \overline{v}_w$). In fact, the optimal reserve price in either auction is strictly below $\overline{v}_w$.

As before, the distributions F_s and F_w are taken to be primitives of the problem. The optimal reserve price in either auction is determined by z and (m_s, m_w). Assume now that $F_w <_{hr} F_s$, so that Proposition 6 can be invoked. Thus, there exists (m_s, m_w) for which the FPA is strictly better than the SPA for all non-prohibitive reserve prices. It follows that the FPA must also be strictly better than the SPA when the reserve price is endogenous and allowed to vary with the auction format.

Proposition 9 *Assume $F_w <_{hr} F_s$ and that $0 \leq z \leq J_s(v)$ for all $v \in [\overline{v}_w, \overline{v}_s]$. Then, there exists (m_s, m_w) such that*

$$\Pi^{FPA}(z, r^{FPA}(z, m_s, m_w), m_s, m_w) > \Pi^{SPA}(z, r^{SPA}(z, m_s, m_w), m_s, m_w).$$

In other words, the SPA is not weakly more profitable than the FPA for all (m_s, m_w).

Proof. See Appendix A. ■

The logic behind Proposition 9 is that the FPA is "almost" efficient when there are many bidders and so $(r, m_s, m_w) \in \mathcal{P}$ for any r that is a candidate for an optimal reserve price in the SPA, given $F_w <_{hr} F_s$. Thus, any claim that the SPA dominates the FPA is sensitive to the seller's own-use valuation and the composition of bidders. In contrast, the example in Section 8 proves that there are (F_s, F_w) for which the FPA is weakly better than the SPA for all (z, m_s, m_w).[9]

In summary, the message is not that the FPA always dominates the SPA, although that is sometimes the case. Instead, the message is that the SPA *cannot* always dominate the FPA; any claim to the contrary is more fragile.

7.3 Small asymmetries

Proposition 9 implies that there is a whole range of own-use valuations for which the FPA strictly dominates the SPA for some (m_s, m_w). A partial converse is pursued in this subsection. Thus, the aim is to establish whether for *any* $(m_s, m_w) \geq (2, 1)$, there exists *some* z such that the FPA strictly dominates the SPA with endogenous reserve prices. Stated differently, a counterpart to Proposition 4 that now allows for endogenous reserve prices is sought.

Unfortunately, a non-trivial issue arises when the reserve price is endogenized. First, to ensure that $(r^{SPA}(z), m_s, m_w) \in \mathcal{P}$ it is necessary that $r^{SPA}(z)$ is "high enough", which generally requires z to be large. However, if z is too large, then it is no longer the case that $r^{SPA}(z) < \overline{v}_w$. Intuitively, $r^{SPA}(z)$ is typically increasing in z because higher z implies that the seller is happier to retain the object. The problem, however, is that as z increases, r^{SPA} may discontinuously jump from some value strictly below $\overline{v}_w$ to some value strictly above $\overline{v}_w$. The reason for the discontinuity is that the seller is either attempting to profit from both groups of bidders by accommodating weak bidders with a reserve price below $\overline{v}_w$, or focusing on extracting as much rent as possible from strong bidders by using a reserve price that is prohibitive for weak bidders. As z increases, the seller switches from the former to the latter approach. Consequently, there are reserve prices close to $\overline{v}_w$ that can never be rationalized in a SPA, regardless of z. Thus, it is hard in general to establish the

[9] See Corollary 2, below, which holds for any $(m_s, m_w) \geq (1, 1)$. The auctions are revenue equivalent if $m_s = 0$ or $m_w = 0$. Thus, the FPA is weakly better than the SPA regardless of (m_s, m_w).

existence of a z for which $(r^{SPA}(z), m_s, m_w) \in \mathcal{P}$.

To overcome this technical difficulty I return to the formulation of the model presented in Section 5.2. Starting from $\overline{v}_s = \overline{v}_w$, it is then possible to consider auctions with "small asymmetries", or, more formally, auctions in which $\overline{v}_s$ is marginally above $\overline{v}_w$.[10] Recall that the example in Section 6 fits this model. In this setting, it can be proven that there are own-use valuations for which the FPA strictly dominates the SPA with endogenous reserve prices.

Proposition 10 *Assume that $F_s(\cdot|\overline{v}_s)$ and $F_w(\cdot)$ satisfy (5)–(6). Then, there is some $\overline{v}_s' > \overline{v}_w$ such that for any $\overline{v}_s \in (\overline{v}_w, \overline{v}_s')$ there exists an own-use valuation z for which*

$$\Pi^{FPA}(z, r^{FPA}(z, m_s, m_w), m_s, m_w) > \Pi^{SPA}(z, r^{SPA}(z, m_s, m_w), m_s, m_w)$$

for all $m_s \geq 2$, $m_w \geq 1$.

Proof. See Appendix A. ■

7.4 Reversals of the profitability ranking

Section 5 assumes that the reserve price is exogenous and the same across auction formats. In this case, whichever auction yields higher expected revenue then also yields higher expected profit. Section 6 establishes that the ranking of the two auctions may change as the reserve price changes. However, Section 7.1 reveals that the optimal (endogenous) reserve price generally differs across auctions. The next example strengthens the conclusion of Section 6 by showing that the profit ranking, even when allowing for endogenous reserve prices, may also flip as the seller's own-use valuation changes.

Example 2 (Sensitivity to the seller's own-use valuation): Consider the following concrete example of the type of model described at the beginning of Example 1. Specifically, assume $F_w(v) = \frac{v}{2}$, $v \in [0, 2]$, and $F_s(v) = \frac{v-1}{\overline{v}_s - 1}$, $v \in [1, \overline{v}_s]$, with $\overline{v}_s \geq 2$. Virtual valuations are $J_w(v) = 2v - 2$ and $J_s(v) = 2v - \overline{v}_s$, respectively. Virtual valuations are strictly increasing and, when $\overline{v}_s = 2$, strictly positive for any $v > \underline{v}_s = 1$. It is then easy to see that the optimal reserve price in the SPA is

[10] Note that a "small asymmetry" refers only to a small distance between $\overline{v}_s$ and $\overline{v}_w$. It is possible that $F_s(v)$ and $F_w(v)$ are "far apart" on $(\underline{v}_s, \overline{v}_w]$.

$r^{SPA}(0) = 1$ when the seller's own-use valuation is zero, or $z = 0$. That is, the good is sold with probability one. In fact, the SPA implements the optimal auction since J_w and J_s coincide. Thus,

$$\Pi^{SPA}(0, r^{SPA}(0, m_s, m_w), m_s, m_w) > \Pi^{FPA}(0, r^{FPA}(0, m_s, m_w), m_s, m_w).$$

By continuity, a small perturbation of F_s, obtained by marginally increasing $\overline{v}_s$, cannot change this ranking when z is held fixed at $z = 0$. However, Proposition 10 proves that there must be some other z for which the FPA is strictly more profitable than the SPA with endogenous reserve prices. ▲

Example 2 establishes that the seller's own-use valuation may be crucial even when selecting among simple auction formats like the SPA and the FPA. This fact represents a challenge to the applied literature where the seller's own-use valuation need not be known. It can perhaps be argued that z can be inferred from the observed reserve price that the seller is using in the real world.[11] Given this premise, however, it may be impossible for the econometrician to determine whether the auction currently in use should be replaced with the alternative auction format. The reason is that the optimal reserve price in the counterfactual auction may be below that used in the real auction. In that case, the econometrician does not access to data that would allow him to calculate optimal profit in the counterfactual auction.

8 Illustration: The uniform model

Assume in this section that both groups of bidders draw types from uniform distributions, or $F_i(v) = \frac{v}{\overline{v}_i}$, $v \in [0, \overline{v}_i]$, $i = s, w$. Maskin and Riley (2000) use this model to illustrate one of their results by utilizing the fact that a closed-form equilibrium solution can be characterized in the two-bidder case. Note that $\frac{\overline{v}_s}{\overline{v}_w}$ can be viewed as a measure of the degree of asymmetry in this model.

A closed-form solution seems unattainable when $m_s > 1$. Nevertheless, Hubbard and Kirkegaard (2015) show that $\widehat{v}$ can be characterized in the absence of a reserve

[11] This is a heroic assumption especially in applications where there is no history of experimentation with the reserve price. After all, determining the optimal reserve price requires knowledge of not only z but also of $F_s(v)$ and $F_w(v)$. Recall also that the optimal reserve price generally depends on m_s and m_w.

price. Appendix B explains how their arguments can be extended to allow for reserve prices. Thus, it is possible to characterize $\widehat{v} = k(\overline{v}_w|r, m_s, m_w)$ for all $r \in [0, \overline{v}_w]$. Moreover, in the uniform model it can also be proven that $(r, m_s, m_w) \in \mathcal{P}$ if and only if $J_w(\overline{v}_w) > J_s(\widehat{v})$. Combining these two observations makes it possible to determine precisely when $(r, m_s, m_w) \in \mathcal{P}$.

To illustrate, fix $\overline{v}_w = 1$. Figure 2 depicts the set of reserve prices and values of $\overline{v}_s$ for which $(r, m_s, m_w) \in \mathcal{P}$, for three possible pairs of (m_s, m_w). In particular, $(r, m_s, m_w) \in \mathcal{P}$ at any $(\overline{v}_s, r)$ point *above* the relevant curve. The fact that the curves shift inwards as m_s and m_w increase reflects Proposition 5.

The approach above makes it possible to handle the case where $m_s \geq 2$. The first result identifies the level of asymmetry required in order to prove that the FPA dominates the SPA regardless of the reserve price. It should be noted that the uniform model satisfies Kirkegaard's (2012a) assumptions. Thus, expected revenue in the FPA is strictly higher than expected revenue in the SPA for any $r \in [0, \overline{v}_w)$ when $m_s = 1$ and $m_w \geq 1$. For this reason, the following results allow for $m_s = 1$.

Corollary 2 *In the uniform model,* $\Delta(r, m_s, m_w) > 0$ *for all* $r \in [0, \overline{v}_w)$ *and all* $m_s \geq 1$ *and* $m_w \geq 1$ *if* $\frac{\overline{v}_s}{\overline{v}_w} > \sqrt{20} - 3 \approx 1.472$. *Thus, if* $\frac{\overline{v}_s}{\overline{v}_w} > \sqrt{20} - 3$, *then*

$$\Pi^{FPA}(z, r^{FPA}(z), m_s, m_w) \geq \Pi^{SPA}(z, r^{SPA}(z), m_s, m_w)$$

for any z *and any* $m_s \geq 1$, $m_w \geq 1$.

Proof. The Corollary is implied by Proposition 13 in Appendix B. ■

The uniform model can also be used to illustrate and strengthen Propositions 9 and 10. As in Proposition 9, the role of the assumption that $\frac{\overline{v}_s}{\overline{v}_w} < 2$ is to ensure that $J_s(v) > 0$ for all $v \in [\overline{v}_w, \overline{v}_s]$.

Proposition 11 *Consider the uniform model and assume that* $1 < \frac{\overline{v}_s}{\overline{v}_w} < 2$. *There exists some* z *for which* $0 < r^{FPA}(z) < r^{SPA}(z) < \overline{v}_w$ *and*

$$\Pi^{FPA}(z, r^{FPA}(z), m_s, m_w) > \Pi^{SPA}(z, r^{SPA}(z), m_s, m_w).$$

for any $m_s \geq 1$, $m_w \geq 1$.

Proof. See Appendix B. ■

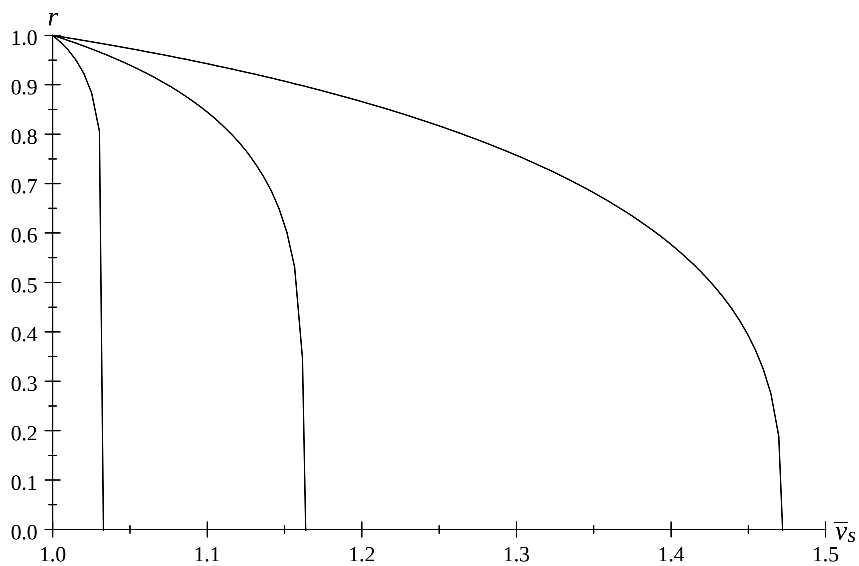

Figure 2: Depicting $\mathcal{P}$. The three curves from right to left describe the cases where (m_s, m_w) equals $(2, 1)$, $(3, 2)$, and $(6, 5)$, respectively.

9 Efficiency

Corollary 1 implies that from an efficiency standpoint a trade-off between the two auctions may exist. The SPA is efficient contingent on a sale, yet it leads to a sale less often than the FPA when $r^{SPA} > r^{FPA}$. It is hard to determine which effect dominates without a closed-form solution of bidding in the FPA. The following example uses related arguments to prove that the FPA may be more efficient.

Example 3 (Efficiency): Assume that $z = 0$. As in Section 8, assume the uniform model applies and that $\overline{v}_w = 1$ and $\overline{v}_s > 2$. Then, $J_s(\overline{v}_w) < z = 0$. Hence, $\Pi^{SPA}(0, r, m_s, m_w)$ attains a local maximum at some $r' > \overline{v}_w$. A reserve price of this magnitude excludes the weak bidders. Standard arguments then lead to the conclusion that r' satisfies $J_s(r') = 0$ or $r' = \frac{1}{2}\overline{v}_s$. However, if $\overline{v}_s$ is not too large, then $\Pi^{SPA}(0, r, m_s, m_w)$ also attains a local maximum at some $r'' \in (0, \overline{v}_w)$. If $m_s = 2$ and $m_w = 3$ then the seller is indifferent between r' and r'' if and only if $\overline{v}_s = \overline{v}'_s = 2.522$; see Figure 3. Assume for now that the parameters $(\overline{v}_s, m_s, m_w)$ take these values. Then, $r' = 1.261$ and $r'' = 0.804$.

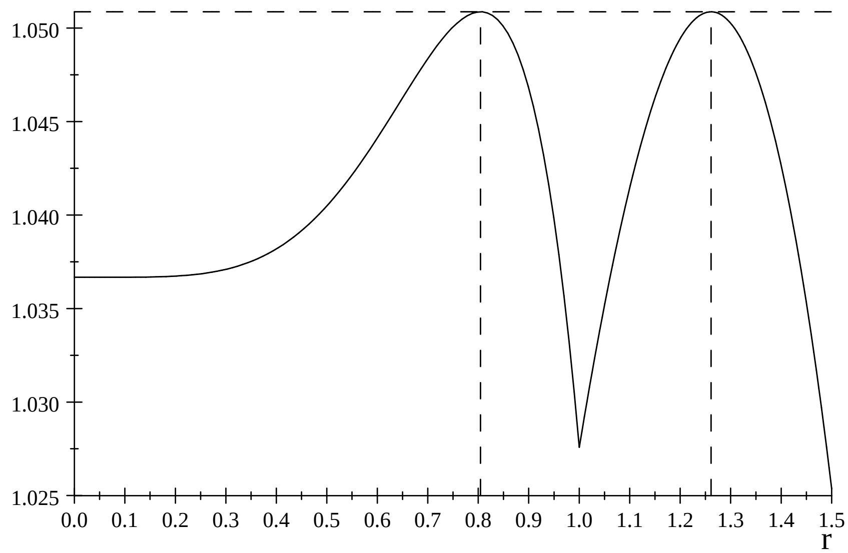

Figure 3: Expected payoff in the SPA.

Now consider the FPA. The FPA is revenue equivalent to the SPA if the seller uses a reserve price above $\overline{v}_w$. Hence, the only candidate for an optimal reserve price in this range is r'. For reserve prices below $\overline{v}_w$, Corollary 2 implies that

$$\max_{r\in(0,\overline{v}_w)} \Pi^{FPA}(0,r,2,3) > \Pi^{SPA}(0,r'',2,3) = \Pi^{SPA}(0,r',2,3) = \Pi^{FPA}(0,r',2,3).$$

Hence, the seller is strictly better off using a FPA with an optimal reserve price below $\overline{v}_w$ than a SPA with reserve price r' or r''.

Next, increase $\overline{v}_s$ marginally above $\overline{v}'_s$. The SPA is now uniquely maximized at a reserve price, r^{SPA}, slightly above r'. If the increase in $\overline{v}_s$ is small enough, however, it cannot change the fact that the optimal reserve price in the FPA, r^{FPA}, remains below $\overline{v}_w$. By continuity, it remains the case that the seller prefers the FPA with the optimal reserve price to the SPA with the optimal reserve price. Moreover, $r^{FPA} < \overline{v}_w < r^{SPA}$.

The characterization of the uniform model detailed in Appendix B reveals that $\widehat{v} = 1.155$ in the FPA when $r = 0$. Proposition 1 then implies that even though r^{FPA} is unknown, $\widehat{v}$ cannot exceed 1.155. Hence, $\widehat{v} < r^{SPA} \approx 1.261$.

It is now possible to compare the efficiency of the two auctions. In the SPA, the object is allocated to the bidder with the highest value, provided this value exceeds r^{SPA}. The allocation in the FPA is efficient if the winner's type is above $\widehat{v}$. Consequently, if the object is sold in the SPA with reserve $r^{SPA} > \widehat{v}$ then it is sold to the exact same bidder in the FPA with reserve r^{FPA}. However, the latter realizes more gains from trade as the object is sold more often.

Bidders with type below r^{FPA} are indifferent between the two auctions since they never win either. Weak bidders with higher types are excluded from the SPA but have a chance of winning the FPA. Hence, they strictly prefer the FPA to the SPA. Moreover, Myerson's (1981) mechanism design arguments reveals that any strong bidder with type above r^{FPA} strictly prefers the FPA to the SPA.[12] It has already been argued that the FPA is strictly more profitable than the SPA. Hence, the seller ex ante strictly prefers the FPA. Thus, the optimal FPA is an ex ante Pareto improvement over the SPA (and even an interim improvement for bidders). ▲

The following proposition records the conclusion of Example 3.

Proposition 12 *The FPA may be ex ante Pareto superior to the SPA when the reserve price is endogenous.*

It is possible that the optimal reserve price equals $\underline{v}_s$ in both auctions. This occurs if $J_i(v) \geq z$ for all $v \in [\underline{v}_s, \overline{v}_i]$ and both $i = s, w$. Since the reserve price is the same in both auctions, the SPA is more efficient than the FPA in this case. Thus, a general and unambiguous ranking of the two auctions in terms of efficiency cannot be obtained in the presence of endogenous reserve prices.

10 Conclusion

Progress on ranking different auctions in the presence of asymmetries has been slow. This paper represents a first step towards understanding the problem when there are more than two bidders. A main simplifying assumption is that any bidder belongs to one of two groups. This assumption is standard in the empirical literature.

The model is otherwise sparsely structured. A key assumption is that the highest possible type is different across bidders. Given that the premise is that bidders are

[12] The expected utility of a strong bidder with type $v \geq r^{FPA}$ in a FPA is $\int_{r^{FPA}}^{v} q_s^{FPA}(x)dx$, where $q_s^{FPA}(x)$ is his winning probability had his type been x. Expected utility in the SPA is $\int_{r^{SPA}}^{v} q_s^{SPA}(x)dx$ for $v \geq r^{SPA}$. The payoff ranking follows from the fact that $q_s^{FPA}(x) > 0 = q_s^{SPA}(x)$ for all $x \in (r^{FPA}, r^{SPA})$ and $q_s^{FPA}(x) = q_s^{SPA}(x)$ for all $x \geq r^{SPA}$.

asymmetric, identical supports would seem to be a knife-edge.[13] The central result is that in any environment consistent with these minimal assumptions, a reserve price exists for which the FPA is strictly more profitable than the SPA. However, the revenue ranking may flip as the reserve price changes. Since the optimal reserve price is endogenous, this last observation puts a renewed emphasis on the seller's own-use valuation. It may play a more central role in selecting the best auction format than suggested by existing theory.

Similarly, which auction is better may depend on the composition of bidders. Unless the asymmetry is large, there always exists an own-use valuation and a bidder composition for which the FPA is strictly more profitable than the SPA. Thus, an impossibility result emerges; the SPA cannot weakly dominate the FPA for all own-use valuations and all combinations of bidders. Likewise, and perhaps contrary to the received wisdom, endogenizing the reserve price may cause the FPA to Pareto dominate the SPA.

These results suggest some caution is prudent when interpreting various findings in the empirical literature. When conducting the counterfactual analysis described in the introduction, it is rare that changes in the reserve price are examined as well. Since the revenue ranking may be sensitive to the reserve price, it may be worthwhile to augment counterfactual studies with a robustness check along this dimension of auction design. More problematically, the best design may depend on the seller's own-use valuation, which is less likely to be known.

The model assumes participation is exogenous, yet it is not without implications for the issue of entry. The value of attracting more participation is well-recognized; see e.g. Bulow and Klemperer (1996). As the revenue ranking may also depend on the composition of bidders, any steps taken to encourage entry should at the very least be accompanied by an examination of whether a change in auction design at the same time is called for. Conversely, the fact that the FPA may be Pareto superior to the SPA for a fixed number of participants suggests that there might be situations where the FPA can attract more bidders than the SPA.

In the two-bidder case, Maskin and Riley (2000) and Kirkegaard (2012a) present stronger conditions under which the FPA is more profitable than the SPA regard-

[13] A common maximal type should thus be assessed based on its economic content. There are situations where it is sensible. For instance, assume bidders are initially symmetric. Then, a subset form a cartel. The cartel's valuation is represented by the maximum of the cartel members' types. In this case, the auction features asymmetric bidders who share a common maximum type.

less of the reserve price. An interesting direction for future research is to examine whether these conditions are sufficient with more bidders as well. The current paper establishes that the same ranking at the very least obtains when the number of bidders is sufficiently large. Thus, the FPA is preferable to the SPA at "corners" of the parameters space, i.e. when the number of bidders is either two or very large.

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Appendix A: Omitted proofs

Descibing the problem: For completeness, I first outline both formulations of the problem. To begin, let $\varphi_i(b)$ denote bidder i's inverse bidding strategy, $b \in [r, \bar{b}_i]$, $i = s, w$. On the range of bids where both groups of bidders are active, $[r, \bar{b}_w]$, $\varphi_w(b)$ and $\varphi_s(b)$ solve the system of differential equations described by

$$\frac{d}{db} \ln F_i(\varphi_i(b)) = \frac{1}{m_s + m_w - 1} \left[\frac{m_j}{\varphi_j(b) - b} - \frac{m_j - 1}{\varphi_i(b) - b} \right], \tag{7}$$

$i, j = s, w$, $i \neq j$, with boundary conditions $\varphi_w(\bar{b}_w) = \bar{v}_w$ and $\varphi_s(\bar{b}_w) = \widehat{v}$. Note that if $\widehat{v} < \bar{v}_s$, then $\varphi_w'(\bar{b}_w) = 0$, by (2). Lebrun (2006) proves that $\varphi_i'(b) > 0$ for all interior bids, however. The solution must also agree with a set of initial conditions, as described in footnote 5.

Second, consider the formulation of the problem in terms $b_w(v)$ and $k(v)$. If his type is v, a weak bidder's problem can be thought of as deciding which type, x, to mimic. His problem is thus to maximize

$$(v - b_w(x)) F_s(k(x))^{m_s} F_w(x)^{m_w - 1}.$$

Similarly, a strong bidder with type $k(v)$ who bids in the common range maximizes

$$(k(v) - b_w(x)) \, F_s(k(x))^{m_s - 1} F_w(x)^{m_w}.$$

By definition of equilibrium, bidders' payoffs are maximized when $x = v$. When $v \in (r, \bar{v}_w)$, the first order conditions yield the system of differential equations

$$\begin{aligned} k'(v) &= \frac{F_s(k(v))}{f_s(k(v))} \frac{f_w(v)}{F_w(v)} T(k(v), b_w(v), v) \\ b_w'(v) &= \frac{f_w(v)}{F_w(v)} (k(v) - b_w(v)) \left[(m_s - 1) \, T(k(v), b_w(v), v) + m_w \right], \end{aligned} \tag{8}$$

where

$$T(k, b_w, v) = \frac{m_w \frac{k - b_w}{v - b_w} - (m_w - 1)}{m_s - (m_s - 1) \frac{k - b_w}{v - b_w}}$$

To compare this formulation of the problem with the previous one, the boundary conditions are that $k(\overline{v}_w) = \widehat{v}$ and $b_w(\overline{v}_w) = \overline{b}_w$.[14] Note that $T(k, b_w, v) \gtreqless 1$ if and only if $k \gtreqless v$. Likewise, holding b_w and v fixed, $T(k, b_w, v)$ is strictly increasing in k. It also holds that $\frac{\partial T(k,b_w,v)}{\partial b_w} \gtreqless 0$ if and only if $k \gtreqless v$. ▲

Proof of Lemma 1. Given these preliminaries it is now possible to prove Lemma 1. Recall that $k(\overline{v}_w) > \overline{v}_w$. To illustrate the proof idea, assume first that the inequality in (1) is strict. Assume there exists some $v_0 \in (r, \overline{v}_w]$ for which $k(v_0) = v_0$. Since $T = 1$ at such a point,

$$k'(v_0) = \frac{F_s(v_0)}{f_s(v_0)} \frac{f_w(v_0)}{F_w(v_0)} < 1.$$

Thus, increasing v beyond v_0 leads to the conclusion that $k(v) \leq v$. However, this contradicts the equilibrium feature that $k(\overline{v}_w) > \overline{v}_w$. The idea is the same when the inequality in (1) is weak. More formally, assume once again that there exists some $v_0 \in (r, \overline{v}_w)$ for which $k(v_0) = v_0$. Based on this "initial condition", the next step is to obtain the solution to the system of differential equations as v increases beyond v_0 (the solution to this initial value problem is unique given the differentiability assumptions imposed on the primitives). To begin, the guess is made that the solution satisfies $k(v) \leq v$ for all $v \geq v_0$. Then, $T \leq 1$, and it follows that

$$\frac{d}{dv} \ln F_s(k(v)) = \frac{f_s(k(v))}{F_s(k(v))} k'(v) \leq \frac{f_w(v)}{F_w(v)} = \frac{d}{dv} \ln F_w(v),$$

independently of $b_w(v)$. By Gronwall's inequality, the solution is then bounded above by the solution that would be obtained if the inequality had been replaced by an equality, or

$$\ln \frac{F_s(k(v))}{F_w(v)} \leq \ln \frac{F_s(v_0)}{F_w(v_0)}.$$

Now, if $k(v) > v$ then, by reverse hazard rate dominance,

$$\frac{F_s(k(v))}{F_w(v)} > \frac{F_s(v)}{F_w(v)} \geq \frac{F_s(v_0)}{F_w(v_0)},$$

which contradicts the previous inequality. Thus, $k(v) \leq v$, thereby verifying the guess made initially. In particular, $k(\overline{v}_w) \leq \overline{v}_w$, but this violates the equilibrium property stated at the beginning of the proof. Hence, there can be no $v_0 \in (r, \overline{v}_w)$ for which

[14] In equilibrium, $k'(v) > 0$ and $b_w'(v) > 0$. Note, however, that if $\widehat{v} < \overline{v}_s$ then $T(k(v), b_w(v), v)$ goes to infinity as v approaches $\overline{v}_s$, by (2).

$k(v_0) = v_0$. By continuity, it then follows that $k(v) > v$ for all $v \in (r, \overline{v}_w]$. ■

Proof of Proposition 1. I first establish the initial conditions are that $b_w(r) = r$ and $k(r) = r$. Lebrun (2006) shows that in general $\varphi_i(r) = r$ for all but at most one bidder i. Stated differently, it is possible that $\varphi_i(r) > r$ for exactly one bidder, such that bidder i has a mass of types that bids r. However, since strategies within any given group is symmetric and $m_s \geq 2$, no strong bidder can bid r for a mass of types. The same holds for weak bidders if $m_w \geq 2$. This leaves the case where $m_w = 1$. Compared to Lebrun (2006), however, here it is assumed that reverse hazard rate dominance applies. By Lemma 1, the weak bidder is more aggressive than the strong bidders, for comparable types. Thus, the weak bidder cannot, in equilibrium, be bidding r for a mass of types. In short, it must hold that $\varphi_i(r) = r$ for all bidders in the current model. Equivalently, the initial conditions to the system in (8) are that $k(r) = r$ and $b_w(r) = r$.

Let $\widehat{v}$ denote the strong bidders' cut-off type and $\overline{b}_w$ the weak bidders' maximum bid when the reserve price is r. Let $\widehat{v}'$ and $\overline{b}_w'$ denote their counterparts when the reserve price increases to r'. Note first that if $\overline{b}_w = \overline{b}_w'$ then $\widehat{v} = \widehat{v}'$, by (2). The system of differential equations are then characterized by the same boundary conditions regardless of whether the reserve price is r or r'. The unique solution in case the reserve is r' must thus coincide with the unique solution when the reserve is r. However, this implies that $b_w(r'|r') = b_w(r'|r) > r'$, which violates the initial conditions of the system when the reserve is r'. Thus, in equilibrium, $\overline{b}_w \neq \overline{b}_w'$.

Consider next the possibility that $\overline{b}_w > \overline{b}_w'$, implying that $\widehat{v}' \geq \widehat{v}$. Assume first that $\widehat{v}' > \widehat{v}$. Hence, for v close to $\overline{v}_w$, $k(v|r')$ is strictly above $k(v|r)$ while $b_w(v|r')$ is strictly below $b_w(v|r)$, or $k(\overline{v}_w|r') = \widehat{v}' > \widehat{v} = k(\overline{v}_w|r)$ and $b_w(\overline{v}_w|r') = \overline{b}_w' < \overline{b}_w = b_w(\overline{v}_w|r)$. Reducing v from $\overline{v}_w$, find the nearest value, v', (if one exists) where one of the new endogenous functions crosses its old counterpart. The argument in the first paragraph rules out that $k(v'|r') = k(v'|r)$ and $b_w(v'|r') = b_w(v'|r)$ at the same time. There are two remaining cases. Assume $b_w(v'|r') = b_w(v'|r)$ but $k(v'|r') > k(v'|r)$. Then, $b_w'(v'|r') > b_w'(v'|r)$, contradicting that $b_w(v|r') < b_w(v|r)$ for $v > v'$. Assume instead that $k(v'|r') = k(v'|r)$ but $b_w(v'|r') < b_w(v'|r)$. Then, $k'(v'|r') < k'(v'|r)$ if $k(v'|r') = k(v'|r) > v'$. However, this contradicts that $k(v|r') > k(v|r)$ for $v > v'$.

Next, assume that $\overline{b}_w > \overline{b}_w'$ but that $\widehat{v}' = \widehat{v}$. This necessitates $\widehat{v}' = \widehat{v} = \overline{v}_s$. It can now be seen that $k(v|r)$ is steeper than $k(v|r')$ near $\overline{v}_w$. Hence, $k(v|r') > k(v|r)$ for v close to, but strictly below, $\overline{v}_w$. By continuity, it is also the case that $b_w(v|r') < b_w(v|r)$

in such a neighborhood. The previous arguments can then be repeated to obtain a contradiction.

Hence, it has now been shown that $\bar{b}_w < \bar{b}'_w$, thereby implying that $\widehat{v}' \leq \widehat{v}$. Stated differently, $b_w(\overline{v}_w|r) < b_w(\overline{v}_w|r')$ and $k(\overline{v}_w|r) \geq k(\overline{v}_w|r')$. Moreover, either $k(\overline{v}_w|r) > k(\overline{v}_w|r')$ or $k(v|r)$ is flatter than $k(v|r')$ near $\overline{v}_w$. In either case, $b_w(v|r) < b_w(v|r')$ and $k(v|r) > k(v|r')$ when v is close to $\overline{v}_w$. Arguments like those above can then be used to prove that these inequalities are unchanged as v is reduced from $\overline{v}_w$ to r'. ∎

Proof of Proposition 2. Consider changes in m_w first. Let $\widehat{v}$ and $\widehat{v}'$ denote the cut-off types when the composition of bidders is (m_s, m_w) and (m_s, m'_w), respectively. Let $\bar{b}_w$ and $\bar{b}'_w$ denote weak bidders' maximum bid in the two environments. Hubbard and Kirkegaard (2015, Proposition 2) have shown that if $\widehat{v} < \overline{v}_s$, as assumed, then $\widehat{v}' < \widehat{v}$.[15] Thus, $k(\overline{v}_w|r, m'_s, m_w) < k(\overline{v}_w|r, m_s, m_w)$. Starting at $\overline{v}_w$, reduce v until the first point is reached (if one exists) where $k(v'|r, m'_s, m_w) = k(v'|r, m_s, m_w) > v'$, with $v' > r$. Assume first that $b_w(v'|r, m'_s, m_w) \geq b_w(v'|r, m_s, m_w)$. Then,

$$\frac{k(\overline{v}_w|r, m'_s, m_w) - b_w(v'|r, m'_s, m_w)}{v' - b_w(v'|r, m'_s, m_w)} \geq \frac{k(\overline{v}_w|r, m_s, m_w) - b_w(v'|r, m_s, m_w)}{v' - b_w(v'|r, m_s, m_w)} > 1.$$

Combined with $m'_w > m_w$ these inequalities ensure that $k'(v'|r, m'_s, m_w) > k'(v'|r, m_s, m_w)$. However, this contradicts the fact that $k(v|r, m'_s, m_w), k(v|r, m_s, m_w)$ at $v > v'$. Assume next that $b_w(v'|r, m'_s, m_w) < b_w(v'|r, m_s, m_w)$. Since $k(v'|r, m'_s, m_w) = k(v'|r, m_s, m_w)$, it must also hold that

$$\begin{aligned} b_s(k(v'|r, m'_s, m_w)|r, m'_s, m_w) &= b_w(v'|r, m'_s, m_w) < b_w(v'|r, m_s, m_w) \\ &= b_s(k(v'|r, m_s, m_w)|r, m_s, m_w), \end{aligned}$$

where the subscript s refers to strong bidder's strategies. Letting $\varphi_i(b|r, m_s, m_w)$ and $\varphi_i(b|r, m'_s, m_w)$ denote the inverse bidding strategy of a bidder in group i, $i = s, w$, there must now exist some b for which $\varphi_s(b|r, m'_s, m_w) > \varphi_s(b|r, m_s, m_w)$ and $\varphi_w(b|r, m'_s, m_w) > \varphi_w(b|r, m_s, m_w)$. However, this is impossible as established in the proof of Hubbard and Kirkegaard's (2015) Proposition 2. Hence, there can be no $v' \in (r, \overline{v}_w)$ for which $k(v'|r, m'_s, m_w) = k(v'|r, m_s, m_w)$. Since $k(\overline{v}_w|r, m'_s, m_w) <$

[15] The statement of Hubbard and Kirkegaard's (2015) result assumes that $m_w \geq 2$. However, as explained earlier, this assumption can be weakened to $m_w \geq 1$ once reverse hazard rate dominance is assumed.

$k(\overline{v}_w|r, m_s, m_w)$, continuity then implies that $k(v|r, m_s', m_w) < k(v|r, m_s, m_w)$ for all $v \in (r, \overline{v}_w]$. The proof of the result for changes in m_s is analogous. ■

Proof of Proposition 3. Lemma 1 establishes the lower bound that $k(v) > v$ for all $v \in (r, \overline{v}_w]$. An upper bound on $k(v)$ is derived next. The proof then concludes by showing that the upper bound converges to v as the number of bidders goes to infinity.

Using (7) and the condition that $\varphi_w'(b) \geq 0$ yield the conclusion that

$$\frac{m_s}{k(v) - b_w(v)} - \frac{m_s - 1}{v - b_w(v)} \geq 0$$

or

$$k(v) \leq \frac{m_s}{m_s - 1} v - \frac{1}{m_s - 1} b_w(v) \tag{9}$$

for all $v \in (r, \overline{v}_w]$. Since $b_w(v)$ is bounded above by v, the last term in (9) goes to zero as $m_s \to \infty$. Since the first term converges to v, it now follows that $k(v) \to v$ as $m_s \to \infty$.

Next, consider changes in m_w instead. In equilibrium, $b_w(v) \leq v$. At the same time, it follows from Myerson (1981) that for any $v \in (r, \overline{v}_w]$,

$$(v - b_w(v)) F_w(v)^{m_w - 1} F_s(k(v))^{m_s} = \int_r^v F_w(x)^{m_w - 1} F_s(k(x))^{m_s} dx$$

or

$$\begin{aligned} b_w(v) &= v - \int_r^v \left(\frac{F_w(x)}{F_w(v)} \right)^{m_w - 1} \left(\frac{F_s(k(x))}{F_s(k(v))} \right)^{m_s} dx \\ &\geq v - \int_r^v \left(\frac{F_w(x)}{F_w(v)} \right)^{m_w - 1} dx \to v \text{ as } m_w \to \infty. \end{aligned}$$

Thus, $b_w(v) \to v$ as $m_w \to \infty$. Once again, (9) now implies that $k(v) \to v$ as $m_w \to \infty$. ■

Proof of Proposition 8. Since

$$\Delta(r, m_s, m_w) = \int_r^{\overline{v}_w} \left(\int_v^{k(v|r)} (J_w(v) - J_s(x)) \, dF_s(x)^{m_s} \right) dF_w(v)^{m_w},$$

the derivative with respect to r is

$$\frac{\partial \Delta(r, m_s, m_w)}{\partial r} = -\left(\int_r^{k(r|r)} (J_w(r) - J_s(x))\, dF_s(x)^{m_s}\right) m_w F_w(r)^{m_w - 1} f_w(r)$$
$$+ \int_r^{\overline{v}_w} \left((J_w(v) - J_s(k(v|r)))\, m_s F_s(k(v|r))^{m_s - 1} f_s(k(v|r)) \frac{\partial k(v|r)}{\partial r}\right) dF_w(v)^{m_w}$$
$$= \int_r^{\overline{v}_w} \left((J_w(v) - J_s(k(v|r)))\, m_s F_s(k(v|r))^{m_s - 1} f_s(k(v|r)) \frac{\partial k(v|r)}{\partial r}\right) dF_w(v)^{m_w},$$

as $k(r|r) = r$. By Proposition 1, $\frac{\partial k(v|r)}{\partial r} < 0$. Next, since $(r', m_s, m_w) \in \mathcal{P}$ it holds that $(r, m_s, m_w) \in \mathcal{P}$ for all $r \in [r', \overline{v}_w)$, by Proposition 5. Hence, for any $r \in [r', \overline{v}_w)$, $J_w(v) - J_s(k(v|r)) > 0$ for all $v \in (r, \overline{v}_w]$. Thus,

$$\frac{\partial \Delta(r, m_s, m_w)}{\partial r} < 0 \text{ for all } r \in [r', \overline{v}_w),$$

which completes the proof. ■

Proof of Corollary 1. By definition,

$$ER^{FPA}(r, m_s, m_w) = ER^{SPA}(r, m_s, m_w) + \Delta(r, m_s, m_w).$$

Likewise, by definition $ER^{SPA}(r^{SPA}(z), m_s, m_w) \geq ER^{SPA}(r, m_s, m_w)$ for any r (including those above $\overline{v}_w$). Since $(r^{SPA}(z), m_s, m_w) \in \mathcal{P}$, $\Delta(r^{SPA}(z), m_s, m_w) > \Delta(r, m_s, m_w)$ for all $r > r^{SPA}(z)$, by Proposition 8. Hence, $ER^{FPA}(r^{SPA}(z), m_s, m_w) > ER^{FPA}(r, m_s, m_w)$ for all $r > r^{SPA}(z)$. Since

$$\frac{\partial ER^{FPA}(r, m_s, m_w)}{\partial r}_{|r = r^{SPA}(z)} = \frac{\partial \Delta(r, m_s, m_w)}{\partial r}_{|r = r^{SPA}(z)} < 0,$$

it immediately follows that $r^{FPA}(z) < r^{SPA}(z)$. The last part of the proposition follows directly from Proposition 8 and the fact that $\Delta(\overline{v}_w, m_s, m_w) = 0$. ■

Proof of Proposition 9. Assume that $0 \leq z \leq J_s(v)$ for all $v \in [\overline{v}_w, \overline{v}_s]$. Then, regardless of (m_s, m_w), the optimal reserve price in either auction is strictly below $\overline{v}_w$. Proposition 6 implies that there exists (m_s, m_w) pairs for which $(r, m_s, m_w) \in \mathcal{P}$ for all $r \in (\underline{v}_s, \overline{v}_w)$. Hence, $\Pi^{FPA}(z, r, m_s, m_w) > \Pi^{SPA}(z, r, m_s, m_w)$ for all $r \in (\underline{v}_s, \overline{v}_w)$. Likewise, by continuity, $\Pi^{FPA}(z, r, m_s, m_w) > \Pi^{SPA}(z, \underline{v}_s, m_s, m_w)$ for some r close to

$\underline{v}_s$. Thus, regardless of what the optimal reserve price is in the SPA, the FPA is even more profitable. ■

Proof of Proposition 10. The proof proceeds in several steps.

STEP 1: Note first that

$$\frac{\partial J_s(v|\overline{v}_s)}{\partial v} = 2 + \frac{g'(v)}{g(v)} \frac{G(\overline{v}_s) - G(v)}{g(v)}.$$

Thus, $J_s(v|\overline{v}_s)$ is strictly increasing in v when v is close to $\overline{v}_s$. Similarly, $J_w(v)$ is strictly increasing in v when v is close to $\overline{v}_w$. By continuity, when $\overline{v}_s$ is close to $\overline{v}_w$ there thus exists some $v' < \overline{v}_w$ such that $J_w(v)$ and $J_s(v|\overline{v}_s)$ are both strictly increasing for all v between v' and $\overline{v}_w$ and $\overline{v}_s$, respectively. Next, recall that $J_s(\overline{v}_w|\overline{v}_s) < J_w(\overline{v}_w)$ whenever $\overline{v}_s > \overline{v}_w$. Thus, there also exists some $v'' < \overline{v}_w$ such that $J_s(v|\overline{v}_s) < J_w(v)$ for all $v \in (v'', \overline{v}_w]$. To clarify, both v' and v'' depend on $\overline{v}_s$. For any $\overline{v}_s$ close to $\overline{v}_w$, consider now the set of types between $\max\{v', v''\}$ and $\overline{v}_w$. For any type, v, in this set, there exists a unique $\tau > v$ that solves $J_w(v) = J_s(\tau|\overline{v}_s)$. Let $\tau(v|\overline{v}_s)$ denote the resulting function. Since $J_w(v)$ and $J_s(v|\overline{v}_s)$ are strictly increasing, $\tau(v|\overline{v}_s)$ is also strictly increasing and differentiable.

Next, note that

$$\frac{\partial \tau(v|\overline{v}_s)}{\partial v} = J_w'(v) \left(\frac{\partial J_s(\tau|\overline{v}_s)}{\partial \tau} \right)^{-1}.$$

Recall that $\tau(\overline{v}_w|\overline{v}_s) = \overline{v}_w$ in the limit where $\overline{v}_s = \overline{v}_w$. Hence, when $\overline{v}_s = \overline{v}_w$

$$\frac{\partial \tau(v|\overline{v}_w)}{\partial v}_{|v=\overline{v}_w} = 1 < \frac{m_s}{m_s - 1}.$$

Thus, for any (m_s, m_w), there is a set of $(v, \overline{v}_s)$, with $v < \overline{v}_w < \overline{v}_s$, close to $(\overline{v}_w, \overline{v}_w)$ for which $\frac{\partial \tau(v|\overline{v}_s)}{\partial v} < \frac{m_s}{m_s-1}$. On this set, $\tau(v|\overline{v}_s)$ is thus bounded below by

$$\underline{\tau}(v|\overline{v}_s) = \tau(\overline{v}_w|\overline{v}_s) + \frac{m_s}{m_s - 1} (v - \overline{v}_w), \tag{10}$$

where $\underline{\tau}(\overline{v}_w|\overline{v}_s) = \tau(\overline{v}_w|\overline{v}_s)$.

Together, (7) and the equilibrium property that $\varphi_w'(b) \geq 0$ imply that

$$\frac{m_s}{k(v) - b_w(v)} - \frac{m_s - 1}{v - b_w(v)} \geq 0$$

or, consistent with (2),

$$k(v) \leq \frac{m_s}{m_s - 1} v - \frac{1}{m_s - 1} b_w(v).$$

Since $b_w(v) \geq r$, $k(v)$ is bounded above by

$$\overline{k}(v) = \frac{m_s}{m_s - 1} v - \frac{1}{m_s - 1} r.$$

Now, since $\underline{\tau}(v|\overline{v}_s)$ and $\overline{k}(v)$ have the same slope, it follows that *if* $\underline{\tau}(\overline{v}_w|\overline{v}_s) > \overline{k}(\overline{v}_w)$ then $\underline{\tau}(v|\overline{v}_s) > \overline{k}(v)$ for all $v \geq r$. In this case,

$$\tau(v|\overline{v}_s) \geq \underline{\tau}(v|\overline{v}_s) > \overline{k}(v|r) \geq k(v|r)$$

and the monotonicity of $J_s(v|\overline{v}_s)$ then implies that $(r, m_s, m_w) \in \mathcal{P}$.

STEP 2: For any $\overline{v}_s$ close to $\overline{v}_w$, define $z(\overline{v}_s) = J_s(\overline{v}_w|\overline{v}_s) > 0$. For a fixed $\overline{v}_s$, assume the seller's own-use valuation is $z(\overline{v}_s)$. Note that $z(\overline{v}_w) = \overline{v}_w$. Thus, the optimal reserve price in either auction is exactly $\overline{v}_w$ when $\overline{v}_s = \overline{v}_w$. Note also that $z'(\overline{v}_s) < 0$. Hence, $z(\overline{v}_s) < \overline{v}_s$ whenever $\overline{v}_s > \overline{v}_w$.

Since $J_s(v|\overline{v}_s)$ is strictly increasing in v when v and $\overline{v}_s$ are close to $\overline{v}_w$, it holds that $z(\overline{v}_s) - J_s(v|\overline{v}_s) < 0$ for all $v > \overline{v}_w$. The implication is that the optimal reserve price in the SPA is strictly below $\overline{v}_w$ whenever $\overline{v}_s$ is above $\overline{v}_w$. At such reserve prices,

$$\begin{aligned}\Pi^{SPA}(z, r, m_s, m_w) \quad = \quad & zF_s(r)^{m_s} F_w(r)^{m_w} + m_w \int_r^{\overline{v}_w} J_w(v) F_w(v)^{m_w - 1} F_s(v)^{m_s} f_w(v) dv \\ & + m_s \int_r^{\overline{v}_w} J_s(v) F_s(v)^{m_s - 1} F_w(v)^{m_w} f_s(v) dv \\ & + m_s \int_{\overline{v}_w}^{\overline{v}_s} J_s(v) F_s(v)^{m_s - 1} f_s(v) dv.\end{aligned}$$

Thus, any optimal reserve price in the SPA, denoted $r(\overline{v}_s)$, must satisfy the first order condition

$$m_s \left[z - J_s(r|\overline{v}_s)\right] \frac{g(r)}{G(r)} + m_w \left[z - J_w(r)\right] \frac{f_w(r)}{F_w(r)} = 0.$$

When $\overline{v}_s = \overline{v}_w$, the first order condition is satisfied at $r(\overline{v}_w) = \overline{v}_w$. By continuity, when $\overline{v}_s$ is marginally above $\overline{v}_w$, $r(\overline{v}_s)$ must remain close to $\overline{v}_w$. Thus, $J_s(v|\overline{v}_s)$ and $J_w(v)$ are strictly increasing in v for all $v \geq r$. Hence, $z(\overline{v}_s) - J_s(r|\overline{v}_s) > z(\overline{v}_s) -$

$J_s(\overline{v}_w|\overline{v}_s) = 0$. To satisfy the first order condition it is then necessary that $z(\overline{v}_s) - J_w(r) < 0$. Consequently, $J_w(r) > z(\overline{v}_s) > J_s(r|\overline{v}_s)$ or $J_w(r) > J_s(\overline{v}_w|\overline{v}_s) > J_s(r|\overline{v}_s)$. Monotonicity then implies that $J_w(v) > J_s(v|\overline{v}_s)$ for all $v \in [r, \overline{v}_w]$. Thus, the analysis in Step 1 is valid. Thus, the last step of the proof is to prove that $\underline{\tau}(\overline{v}_w|\overline{v}_s) > \overline{k}(\overline{v}_w)$ when $\overline{v}_s$ is marginally above $\overline{v}_w$.

STEP 3: Given $z(\overline{v}_s) = J_s(\overline{v}_w|\overline{v}_s)$, it is straightforward to show that when $\overline{v}_s = \overline{v}_w$,

$$r'(\overline{v}_w) = -\frac{m_w}{2}\frac{1}{m_s \frac{g(\overline{v}_w)}{G(\overline{v}_w)}\frac{F_w(\overline{v}_w)}{f_w(\overline{v}_w)} + m_w} \geq -\frac{1}{2}\frac{m_w}{m_s + m_w} > -\frac{m_s - 1}{2}, \tag{11}$$

where the first inequality comes from reverse hazard rate dominance and the second inequality from the fact that $m_s \geq 2$. Given the optimal reserve price in the SPA, write the bound on $k(v)$ as

$$\overline{k}(v|r(\overline{v}_s)) = \frac{m_s}{m_s - 1}v - \frac{1}{m_s - 1}r(\overline{v}_s),$$

with

$$\frac{\partial \overline{k}(\overline{v}_w|r(\overline{v}_s))}{\partial \overline{v}_s}_{|\overline{v}_s = \overline{v}_w} = -\frac{1}{m_s - 1}r'(\overline{v}_s) < \frac{1}{2},$$

by (11).

In contrast,

$$\frac{\partial \tau(v|\overline{v}_s)}{\partial \overline{v}_s} = \frac{g(\overline{v}_s)}{g(v)}\left(\frac{\partial J_s(\tau|\overline{v}_s)}{\partial \tau}\right)^{-1}.$$

Evaluated at $\tau(\overline{v}_w|\overline{v}_s)$, the term in the parenthesis reduces to 2 when $\overline{v}_s = \overline{v}_w$. Hence, by (10)

$$\frac{\partial \underline{\tau}(\overline{v}_w|\overline{v}_s)}{\partial \overline{v}_s}_{|\overline{v}_s = \overline{v}_w} = \frac{\partial \tau(\overline{v}_w|\overline{v}_s)}{\partial \overline{v}_s}_{|\overline{v}_s = \overline{v}_w} = \frac{1}{2}.$$

Since $\underline{\tau}(\overline{v}_w|\overline{v}_w) = \overline{v}_w = \overline{k}(\overline{v}_w|r(\overline{v}_w))$, it follows that $\underline{\tau}(\overline{v}_w|\overline{v}_s) > \overline{k}(\overline{v}_w|r(\overline{v}_s))$ when $\overline{v}_s$ is marginally above $\overline{v}_w$. By the argument in the Step 1 (which is valid by Step 2), $(r(\overline{v}_s), m_s, m_w) \in \mathcal{P}$ when $\overline{v}_s$ is marginally above $\overline{v}_w$. The Proposition now follows by invoking Corollary 1. ■

Appendix B: Details of the uniform model

In general, two practical issues arise when checking whether $(r, m_s, m_w) \in \mathcal{P}$. First, there is the issue of characterizing equilibrium in the FPA, as summarized by the function $k(v)$. Second, a continuum of inequalities must then be checked to verify that $(r, m_s, m_w) \in \mathcal{P}$. The second problem is much reduced in the uniform model, however.

Lemma 2 *In the uniform model,* $(r, m_s, m_w) \in \mathcal{P}$ *if and only if* $J_w(\overline{v}_w) > J_s(\widehat{v})$ *or, equivalently, if* $\widehat{v} < \frac{1}{2}(\overline{v}_s + \overline{v}_w)$.

Proof. In the uniform model, virtual valuations take the simple form $J_i(v) = 2v - \overline{v}_i$. Note that virtual valuations are strictly increasing in v. Let $\tau(v)$ denote the unique solution to $J_s(\tau) = J_w(v)$, or $\tau(v) = v + \frac{1}{2}(\overline{v}_s - \overline{v}_w)$. Thus, $(r, m_s, m_w) \in \mathcal{P}$ if $k(v) < \tau(v)$ for all $v \in (r, \overline{v}_w]$. Note that $\tau'(v) = 1$. If it is ever the case that $k(v) = \tau(v)$, then

$$k'(v) > \frac{F_s(\tau(v))}{f_s(\tau(v))} \frac{f_w(v)}{F_w(v)} = \frac{\tau(v)}{v} > 1.$$

Hence, $k(v)$ and $\tau(v)$ can cross at most once. Therefore, $(r, m_s, m_w) \in \mathcal{P}$ if and only if $k(\overline{v}_w) < \tau(\overline{v}_w)$, or, stated differently, if and only if $\widehat{v} < \frac{1}{2}(\overline{v}_s + \overline{v}_w)$. ■

Lemma 2 implies that it is not necessary to characterize the function $k(v)$ in its entirety. Knowing $k(\overline{v}_w)$ (i.e., $\widehat{v}$) is sufficient to establish whether $(r, m_s, m_w) \in \mathcal{P}$. Now, assuming the reserve price is zero, Hubbard and Kirkegaard (2015) characterizes the equilibrium values of $\widehat{v}$ and $\overline{b}_w$. The latter can be written as

$$\widehat{v} = \min\{\overline{v}_s, \alpha(m_s, m_w)\overline{v}_w\},$$

where, when $m = m_s + m_w$,

$$\alpha(m_s, m_w) = \frac{2m_s m_w - (m_s + 1)(m - 1) + \sqrt{(m_s + 1)^2 (m - 1)^2 - 4m_w m_s (m - 1)}}{2m_w (m_s - 1)}.$$

Hence, bid-separation occurs if and only if $\frac{\overline{v}_s}{\overline{v}_w} > \alpha(m_s, m_w)$. If $(m_s, m_w) = (2, 1)$, bid-separation occurs if and only if $\overline{v}_s$ is at least 23.61% larger than $\overline{v}_w$. If $(m_s, m_w) = (3, 2)$, the corresponding number is 8.11%. This provides an illustration of how much more likely bid-separation becomes when the number of bidders increases. Example 3 in Section 9 relies on the fact that $\alpha(2, 3) = 1.155$.

If the reserve price is set to zero, the condition in Lemma 2 holds if and only if

$$\frac{\overline{v}_s}{\overline{v}_w} > 2\alpha(m_s, m_w) - 1.$$

Thus, if $(m_s, m_w) = (2, 1)$ and $\frac{\overline{v}_s}{\overline{v}_s} > 1.472$ – or $\overline{v}_s$ is at least 47.2% larger than $\overline{v}_w$ – then bid-separation is automatically so severe that $(r, m_s, m_w) \in \mathcal{P}$ for all $r \in (0, \overline{v}_w)$ and all $m_s \geq 2$ and $m_w \geq 1$. In this case, then, the FPA is strictly more profitable than the SPA for any $r \in (0, \overline{v}_w)$. Indeed, since $\Delta(r, m_s, m_w)$ is decreasing in r on this interval, it also holds that the FPA is strictly more profitable than the SPA at $r = 0$. Corollary 2 in Section 8 restates the last part of Proposition 13 and adds the observation that the result extends to the $m_s = 1$ case, as demonstrated by Kirkegaard (2012a).

Proposition 13 *In the uniform model, $\Delta(r, m_s, m_w) > 0$ for all $r \in [0, \overline{v}_w)$ if $\frac{\overline{v}_s}{\overline{v}_w} > 2\alpha(m_s, m_w) - 1$. This condition is satisfied for any $m_s \geq 2$ and $m_w \geq 1$ if $\frac{\overline{v}_s}{\overline{v}_s} > \sqrt{20} - 3 \approx 1.472$.*

A main point of the paper is that a reserve price allows one to manipulate equilibrium to trigger bid-separation. That is, it is possible to engineer k to satisfy $\widehat{v} = k(\overline{v}_w) \leq \tau(\overline{v}_w)$ even if the asymmetry is smaller than assumed in Proposition 13. From (2), it is possible to infer which $\overline{b}_w$ is required to obtain a target value of $\widehat{v}$. The next step is then to manipulate r until this is achieved. As mentioned above, Hubbard and Kirkegaard (2015) characterize the equilibrium value of $(\widehat{v}, \overline{b}_w)$ in the uniform model when r is fixed at zero. Their logic extends to $r > 0$. Given a reserve price of r, the probability that the good is sold is $(1 - F_s(r)^{m_s} F_w(r)^{m_w})$. Hubbard and Kirkegaard (2015) derive another expression of the probability of sale in the FPA. This expression, which also holds for any $r \in [0, \overline{v}_w)$, can be reduced to

$$1 - \left(\frac{\widehat{v}}{\overline{v}_s}\right)^{m_s - 1} \left(m_s \frac{\overline{b}_w}{\overline{v}_s} - \frac{\widehat{v}}{\overline{v}_s}\left[m_s + m_w - 1 - m_w \frac{\overline{b}_w}{\overline{v}_w}\right]\right).$$

Setting the two expressions equal to each other yields

$$F_s(r)^{m_s} F_w(r)^{m_w} = \left(\frac{\widehat{v}}{\overline{v}_s}\right)^{m_s - 1} \left(m_s \frac{\overline{b}_w}{\overline{v}_s} - \frac{\widehat{v}}{\overline{v}_s}\left[m_s + m_w - 1 - m_w \frac{\overline{b}_w}{\overline{v}_w}\right]\right)$$

or

$$r = \left[\overline{v}_w^{m_w} \widehat{v}^{m_s - 1} \left(m_s \overline{b}_w - \widehat{v} \left[m_s + m_w - 1 - m_w \frac{\overline{b}_w}{\overline{v}_w} \right] \right) \right]^{\frac{1}{m_s + m_w}} \tag{12}$$

Given a target for $\widehat{v}$, $\overline{b}_w = m_s \overline{v}_w - (m_s - 1)\,\widehat{v}$ is as mentioned determined from (2). Then, (12) uniquely identifies the value of r that is required to obtain $\widehat{v}$. Note that this is independent of $\overline{v}_s$ for a fixed value of $\widehat{v}$, at least as long as $\overline{v}_s > \widehat{v}$. The reason is that the system of differential equations describing behavior at bids below $\overline{b}_w$ is unaffected by $\overline{v}_s$; the reverse hazard rate, $\frac{f_s(v)}{F_s(v)}$, is independent of $\overline{v}_s$.

The optimal reserve price generally varies with the auction format. An exception arises when the optimal reserve price is above $\overline{v}_w$, such that weak bidders are excluded. Then, regardless of the auction format, expected profit is

$$zF_s(r)^{m_s} + \int_r^{\overline{v}_s} J_s(v) dF_s(v)^{m_s}. \tag{13}$$

The derivative with respect to r is proportional to $z - J_s(r)$. Since $J_s(v)$ is strictly increasing, the expression in (13) is single-peaked in r on $r \in [\overline{v}_w, \overline{v}_s]$. At $r = \overline{v}_w$, the derivative equals $z - J_s(\overline{v}_w) = z + \overline{v}_s - 2\overline{v}_w$. To continue, assume $z - J_s(\overline{v}_w) < 0$, or $z + \overline{v}_s \leq 2\overline{v}_w$. Then, the optimal reserve price in either auction must be strictly below $\overline{v}_w$. It can now easily be verified that the optimal reserve price in the SPA is

$$r^{SPA}(z) = \frac{m_w + m_s \overline{v}_s}{2(m_w + m_s)} + \frac{z}{2}.$$

Figure 4 assumes $(m_s, m_w) = (2, 1)$ and superimposes $r^{SPA}(\frac{1}{2})$ and $r^{SPA}(0)$ on top of the relevant curve from Figure 2. The figure thus reveals that Corollary 1 applies if $z = 0$ whenever $\overline{v}_s > 1.3788$. If $z = \frac{1}{2}$ then it applies whenever $\overline{v}_s > 1.238$. In fact, as stated in Proposition 11 in Section 8, it can be shown that there always exists some z for which Corollary 1 applies when $\overline{v}_s < 2$.

A sketch of the proof of Proposition 11 follows. As above, normalize $\overline{v}_w = 1$. Thus, $\overline{v}_s \in (1, 2)$. Assume first that $m_s \geq 2$. Next, fix $z = J_s(\overline{v}_w) = 2 - \overline{v}_s \in (0, 1)$. By design, $z - J_s(\overline{v}_w) = 0$, which implies that the optimal reserve price in either auction is strictly below $\overline{v}_w = 1$. Note that $r^{SPA}(z) = r^{SPA}(2 - \overline{v}_s)$ is linear in $\overline{v}_s$, with derivative $-\frac{m_w}{2(m_w + m_s)}$. Moreover, at the corner where $\overline{v}_s = 1$, $r^{SPA}(1) = 1$ (since in this case the seller values the object more highly than the buyers do). Thus, at the corner, $r^{SPA}(2 - \overline{v}_s)$ coincides with the downwards sloping curve Figure 2.

The latter can be shown to be concave in $\overline{v}_s$. Moreover, at $\overline{v}_s = 1$, the concave curve decreases faster than the linear function $r^{SPA}(2-\overline{v}_s)$; its slope at that point is $-\frac{m_s-1}{2} < -\frac{m_w}{2(m_w+m_s)}$. Hence, $r^{SPA}(2-\overline{v}_s)$ is strictly above the concave curve for any $\overline{v}_s > 1$. Thus, by construction, there exists some z for which $(r^{SPA}(z), m_s, m_w) \in \mathcal{P}$. The proposition now follows by invoking Corollary 1. If $m_s = 1$, the proposition follows from Kirkegaard (2012a).

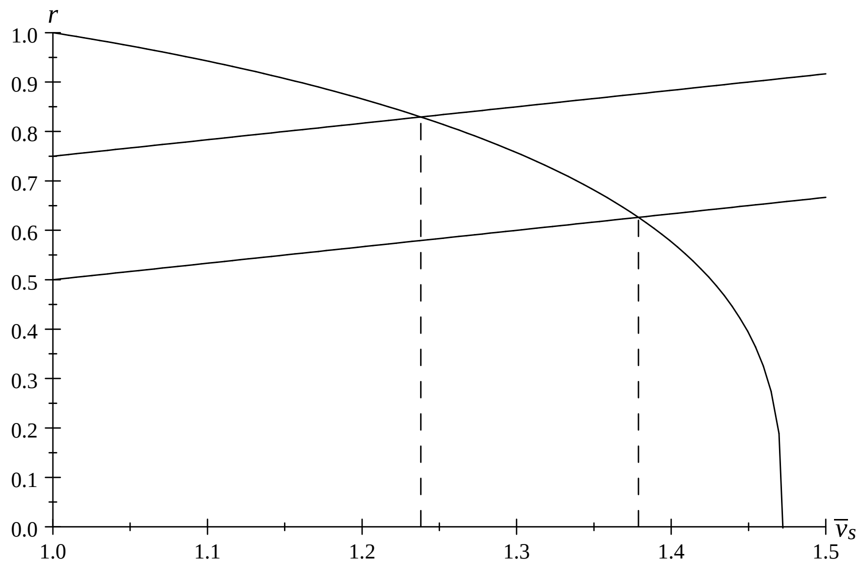

Figure 4: The highest (lowest) upwards sloping cuve depicts $r^{SPA}(\frac{1}{2})$ $(r^{SPA}(0))$ as a function of $\overline{v}_s$.

Appendix C: Extensions

C.1 Auctions with one strong bidder

Kirkegaard's (2012a) approach accommodates any $m_w \geq 1$. However, it necessitates that $m_s = 1$. Moreover, he imposes stronger assumptions on the relationship between F_s and F_w. This subsection establishes that these additional assumptions are not required in order to extend Proposition 4 to the $m_s = 1$ case. Thus, I will assume only that *(i)* $\overline{v}_s > \overline{v}_w$, *(ii)* F_s dominates F_w in terms of the reverse hazard rate, and, for expositional simplicity, that *(iii)* $J_s(v)$ is strictly increasing.

Bid-separation never arises when there is just one strong bidder. Thus, $k(\overline{v}_w) = \widehat{v} = \overline{v}_s$ is the same regardless of the reserve price. However, it is easy to see from the system in (8) that $\overline{b}_w$ is strictly increasing in r. This in turn means that $k'(\overline{v}_w)$

becomes larger and larger as r increases. Since $\overline{b}_w > r$, it also holds that $\overline{b}_w$ converges to $\overline{v}_w$ as r converges to $\overline{v}_w$. Thus, from (8), $k'(\overline{v}_w)$ can be made arbitrarily large simply by selecting a reserve price that is sufficiently close to $\overline{v}_w$.

When the weak bidders' type, v, is sufficiently high – such that $J_w(v) > J_s(v)$ – there must exist some $\kappa > v$ for which

$$\int_v^{\kappa(v)} (J_w(v) - J_s(x))\, dF_s(x) = 0.$$

The assumption that $J_s(v)$ is strictly increasing implies that $\kappa(v)$ is unique and that

$$\int_v^{k(v|r,m_s,m_w)} (J_w(v) - J_s(x))\, dF_s(x) > 0$$

as long as $k(v) \in (v, \kappa(v))$. It can be verified that $\kappa(\overline{v}_w) = \overline{v}_s$. Now, $\kappa(v)$ is independent of r, whereas $k(v)$ depends on r. Moreover, $k(\overline{v}_w) = \kappa(\overline{v}_w)$. Since $k'(\overline{v}_w)$ can be made arbitrarily large by letting r converge to $\overline{v}_w$, it now follows that there exists large r for which

$$k(v|r, m_s, m_w) \in (v, \kappa(v)) \text{ for all } v \in (r, \overline{v}_w)\,.$$

By (3), the FPA dominates the SPA at such a reserve price. Note that Example 1 in Section 6 where the revenue ranking is reversed as r increases can then also be made to work in a model with just two bidders.

C.2 Weaker sufficient conditions

Assume again that $m_s \geq 2$. As emphasized earlier, the condition that $(r, m_s, m_w) \in \mathcal{P}$ is sufficient but not necessary to conclude that $\Delta(r, m_s, m_w) > 0$. For instance, the weaker condition that (r, m_s, m_w) belongs to

$$\widehat{\mathcal{P}} = \left\{ (r, m_s, m_w) \middle| \int_v^{k(v|r,m_s,m_w)} (J_w(v) - J_s(x))\, dF_s(x)^{m_s} > 0 \text{ for all } v \in (r, \overline{v}_w] \right\}$$

is sufficient to obtain the same ranking, as can be seen from (3). Replacing $\mathcal{P}$ by $\widehat{\mathcal{P}}$ is analogous to how Kirkegaard (2012a) refines Maskin and Riley's (2000) mechanism design argument.

Recall that $J_w(v) > J_s(x)$ when v and x are both close to $\overline{v}_w$. Thus,

$$\int_v^k (J_w(v) - J_s(x))\, dF_s(x)^{m_s} \tag{14}$$

is positive if v and k are close to $\overline{v}_w$. Consequently, when r is close to $\overline{v}_w$, $(r, m_s, m_w) \in \widehat{\mathcal{P}}$. Thus, a counterpart to Proposition 4 exists in which $\mathcal{P}$ is replaced by $\widehat{\mathcal{P}}$.

However, the results on optimal reserve prices – like Proposition 8 – are harder to extend. Proposition 8 is valid because a decrease in k is unambiguously desirable when $(r, m_s, m_w) \in \mathcal{P}$. However, this may not be the case when $(r, m_s, m_w) \in \widehat{\mathcal{P}}$.

The uniform model can be used to illustrate the gain in moving from $\mathcal{P}$ to $\widehat{\mathcal{P}}$. Recall that $J_s(v)$ is monotonic in this model. Moreover,

$$k'(v) > \frac{F_s(k(v))}{f_s(k(v))} \frac{f_w(v)}{F_w(v)} = \frac{k(v)}{v},$$

or

$$\frac{d}{dv} \ln \frac{k(v)}{v} > 0.$$

Given $\frac{k(v)}{v} = \frac{\widehat{v}}{\overline{v}_w}$ at $v = \overline{v}_w$, the above bound on the slope of $\frac{k(v)}{v}$ now yields the conclusion that

$$\frac{k(v)}{v} < \frac{\widehat{v}}{\overline{v}_w} \text{ for all } v \in [r, \overline{v}_w).$$

In other words, an upper bound on $k(v)$ has been obtained, with $k(v) \leq \frac{\widehat{v}}{\overline{v}_w} v$ for all $v \in [r, \overline{v}_w]$. Thus, for any $v \in [r, \overline{v}_w)$, (14) is strictly positive if

$$\int_v^{\frac{\widehat{v}}{\overline{v}_w} v} (J_w(v) - J_s(x))\, dF_s(x)^{m_s} \geq 0. \tag{15}$$

It is not hard to verify that the derivative of the expression on the left with respect to v is either positive for all v or first positive and then negative. Hence, if the condition in (15) is satisfied at the endpoints, $v \in \{0, \overline{v}_w\}$, then the condition is also satisfied in the interior. Since the condition is trivially satisfied at $v = 0$, the conclusion emerges that (15) is satisfied for all v if and only if it is satisfied at $v = \overline{v}_w$, where it of course reduces to

$$\int_{\overline{v}_w}^{\widehat{v}} (J_w(\overline{v}_w) - J_s(x))\, dF_s(x)^{m_s} \geq 0.$$

It is now straightforward to check if this condition is satisfied for any given $\widehat{v}$. Note that if the condition is satisfied for some value of $\widehat{v}$ then it is also satisfied for all lower values, keeping in mind that $\widehat{v} \geq \overline{v}_w$.

To illustrate, normalize $\overline{v}_w = 1$ and assume for concreteness that $m_s = 2$, $m_w = 1$. Then, $(r, m_s, m_w) \in \widehat{\mathcal{P}}$ if and only if

$$\widehat{v} < \widehat{v}^c(\overline{v}_s) = \frac{1}{8}\left((3\overline{v}_s - 1) + \sqrt{3\,(3\overline{v}_s - 1)\,(\overline{v}_s + 5)}\right).$$

This critical value of $\widehat{v}$ is extremely close to $\overline{v}_s$. For instance, $\widehat{v}^c(1.5) = 1.47$. Thus, minimal bid-separation is required for $(r, m_s, m_w) \in \widehat{\mathcal{P}}$. If $\overline{v}_s > 1.244$ then $\widehat{v}$ can be shown to exceed $\widehat{v}^c(\overline{v}_s)$ regardless of the reserve price. In comparison, from Section 8, bid-separation becomes an equilibrium feature for any r as soon as $\overline{v}_s > 1.236$.

Following the same steps as in Section 8, it is possible to determine for which r it holds that $\widehat{v} < \widehat{v}^c(\overline{v}_s)$ and thus $(r, m_s, m_w) \in \widehat{\mathcal{P}}$. This is depicted in Figure 5 along with the curve from Figure 2 that defines when $(r, m_s, m_w) \in \mathcal{P}$. Specifically, $(r, m_s, m_w) \in \widehat{\mathcal{P}}$ in the area above the lowest curve, while $(r, m_s, m_w) \in \mathcal{P}$ in the area above the highest curve. The area between the curves thus represents how much is gained by replacing $\mathcal{P}$ with $\widehat{\mathcal{P}}$.

The inner integral in (3) is positive for all v if and only if $(r, m_s, m_w) \in \widehat{\mathcal{P}}$. This property is also sufficient but not necessary for the FPA to outperform the SPA. Weakening the sufficient conditions would thus require one to handle cases where the inner integral in (3) changes sign. This is left for future research.

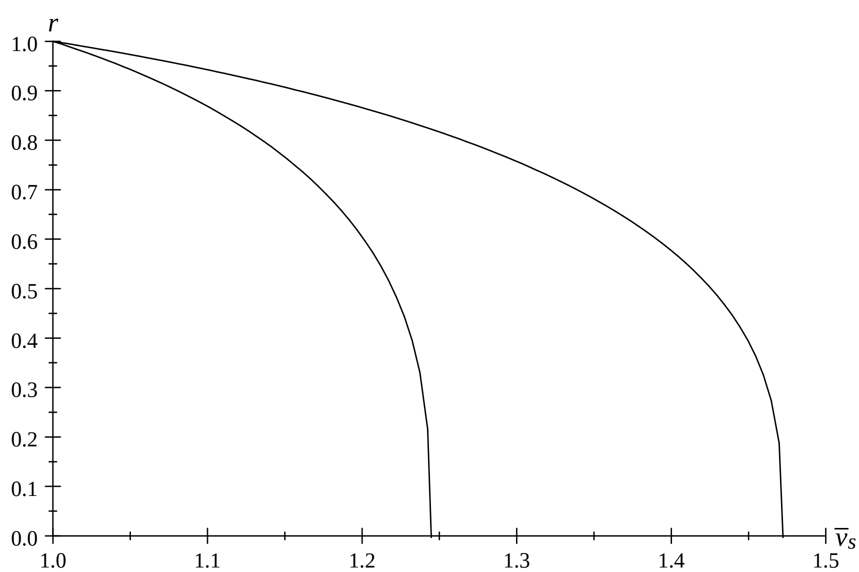

Figure 5: Comparing $\mathcal{P}$ and $\widehat{\mathcal{P}}$ for $m_s = 2$, $m_w = 1$.