Asymmetric First Price Auctions*

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Abstract

We propose a new approach to asymmetric first price auctions which circumvents having to directly examine bidding strategies. Specifically, the ratio of bidders' payoffs is compared to the ratio of the distribution functions that describe beliefs. This comparison allows a number of easy inferences. In the existing theoretical literature, assumptions of first order stochastic dominance or stronger imply that the latter ratio has very specific properties. Most existing results therefore follow as simple corollaries from our two main results. We prove that first order stochastic dominance is necessary for bidding strategies *not* to cross. When this assumption is relaxed in the numerical literature it is done in a manner that leads to exactly one crossing. We construct examples with several crossings. General results are provided for types of asymmetry not studied before, including second order stochastic dominance. In this case, the bid distributions will cross in auctions with two bidders.

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1 Introduction

Though the rules of a first price auction are simple, the same cannot necessarily be said for equilibrium behavior, especially when bidders are asymmetric. In fact, the existence of an equilibrium was not established until recently.[1] Since then, a few properties of bidding behavior have been uncovered, including, for example, the tendency for weak bidders to bid more aggressively than strong bidders.[2]

When analyzing first price auctions it is natural to start by examining bidding strategies directly, as is done in the existing literature. However, this approach is made difficult by the fact that equilibrium bidding strategies are derived from a system of differential equations (arising from the first order conditions), which generally escapes explicit solution. In this paper, we will sidestep these difficulties by exploring another method of analyzing asymmetric first price auctions.[3] As outlined in an earlier version of the paper, Kirkegaard (2006), a similar method can be used to analyze asymmetric all-pay auctions as well.

Comparing bidders pairwise, we suggest focusing on the ratio of the two bidders' expected payoffs, which is endogenous. It turns out that it is straightforward to relate this ratio to the ratio of the distribution functions that describe bidders' beliefs, where the latter represents the primitives of the model. This comparison leads to insights on how bidders' payoffs are ranked and on how bidding strategies relate. Moreover, we use arguments from mechanism design theory to establish that once bidders' payoffs have been ranked several useful inferences concerning winning probabilities and, from there, the ex ante distributions of bids, can be made.

Intuitively, the ratio of the distribution functions is a natural measure of the

[1] See Lebrun (1999) and Maskin and Riley (2000b). Several papers contain examples in which explicit derivations of bidding strategies are possible. See Vickrey (1961), Greismer et al (1967), Plum (1992), Cheng (2006), and Kaplan and Zamir (2007). Riley and Samuelson (1981) derived bidding strategies in the symmetric case. Marshall et al (1994) use numerical methods to estimate bidding strategies in asymmetric auctions. Numerical methods are also used in e.g. Bajari (2001), Gayle and Richard (2005), and Li and Riley (2006).

[2] See Lebrun (1999), Maskin and Riley (2000a), and, for a related point, Fibich et. al. (2002). Maskin and Riley (2000a) also show that there is no unambiguous ranking in terms of revenue of the first and second price auctions. Lebrun (1998) examines the consequences on revenue and bidding when one buyer becomes stronger, while Cantillon (2004) compares revenue for different degrees of asymmetry among buyers. Fibich and Gavious (2003) use perturbation methods to study asymmetries. See Krishna (2002) for an introduction to asymmetric auctions.

[3] Milgrom (2004) and Hopkins (2007) consider another method of analysis. Milgrom (2004) reproduces existing results, while Hopkins (2007) adds a new result (and, in addition, examines all-pay auctions). We discuss the latter in Section 6. See Cheng and Tan (2007) for an application of this method to an asymmetric common value auction.

relative strength of the two bidders. The *level* of the ratio (whether it is above or below one) reveals which bidder is *stronger*, i.e. who is more likely to have a higher valuation. Likewise, the *slope* of the ratio shows how a bidder's strength is *changing* compared to his rival as the valuation increases. It is perhaps not surprising that bidding behavior in the auction is determined by the relative strength of the bidders in combination with how the relative strength evolves. In fact, the two main results prove that the shape of the ratio of the distribution functions, as summarized by the number of times it equals one and the number of times the slope changes sign, provides valuable information on how payoffs and bidding strategies compare over the set of valuations.

The standard assumption in the existing theoretical literature on asymmetric auctions is that the distributions that characterize bidders can be ordered according to first order stochastic dominance. Equivalently, the ratio of the distribution function of two bidders is globally above or below one. To further refine the results, the stronger assumption of reverse hazard rate dominance is also often imposed. The latter assumption is equivalent to the assumption that the ratio of the distribution functions is monotonic. Thus, the standard assumptions imply the ratio of the distribution functions have very specific properties. In contrast, the method proposed here allows us to obtain a number of results for arbitrary shapes of this ratio. In other words, we are not constrained to the usual type of asymmetry.

In summary, the contribution of the paper is twofold. First, a new approach to asymmetric first price auctions is proposed. Second, this paper appears to be the first to *systematically analyze* bidding behavior in situations of asymmetry not described by first order stochastic dominance, or stronger. Hence, several new results are also presented.

However, the approach taken here highlights the consequences of the standard assumptions. Thus, to demonstrate the usefulness of the approach, we reiterate many of the most important existing results, and offer new and simpler proofs of these. Hence, the paper also serves to synthesize and unify the literature on asymmetric first price auctions. For instance, the winning probabilities mentioned earlier will inform us of bidders' preferences for different auction formats, and the type of competition they face. Incidentally, we would argue the winning probabilities are interesting in their own right, yet the standard approach reveals little about these.

One of the most significant results of the current literature is that reverse hazard rate dominance is sufficient to get behavior where one bidder bids consistently more aggressively than another. In this paper we add the complementary result that first order stochastic dominance is *necessary* for this outcome. Kaplan and Zamir (2007) have provided an analytical example of an asymmetric auction where bidding strategies cross. Gayle and Richard (2005) supply a numerical example with the same property. The common feature is that first order stochastic dominance is violated.[4]

In fact, we show that the number of times bidding strategies cross is bounded above by the number of stationary points of the ratio of the distribution functions. We also establish sufficient conditions under which the number of crossings equals this upper bound. Moreover, all existing classes of numerical examples are shown to imply ratios with at most one peak. Thus, with the current specifications, the numerical literature will be unable to provide examples with several crossings. However, we show that it is straightforward to construct examples in which bidding strategies cross several times.

The other existing major result is that first order stochastic dominance imply that the equilibrium bid distributions are also characterized by first order stochastic dominance. In other words, if the distributions from which valuations are drawn do not cross, then bid distributions will not cross either.

Arguably, the natural next step would be to examine second order stochastic dominance. Notice that the two distribution functions will cross if one distribution second order stochastically dominates the other, but first order stochastic dominance does not apply. For example, this holds if one distribution function is a mean preserving spread over the other, which describes a situation where one bidder's valuation is less "predictable" than that of the competition. In the two-bidder case, we show that second order stochastic dominance without first order stochastic dominance imply that bid distributions cross. We provide conditions under which they cross exactly once.[5]

We consider the proposed method of analysis to be complementary to the standard approach. Indeed, in some cases we combine the two methods. Though we provide new results as well as new proofs of many existing results, we ignore the important questions of existence, uniqueness and revenue comparison of different auction formats. These difficult questions have been tackled by Lebrun (1999, 2006) and Maskin and Riley (2000a, 2000b, 2003).

The paper is organized as follows. The model is presented in Section 2, and some preliminary results on how payoffs, bids, winning probabilities and bid distributions relate are offered. In Section 3, the core of the paper, bidders are compared pairwise,

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[4] Indeed, Maskin and Riley (2000a, footnote 14)) contain an example demonstrating that bidding strategies may cross even when first order stochastic dominance is satisfied. We explain this result in Section 3, where we consider a class of situations encompassing Maskin and Riley's (2000a) example. In conclusion, first order stochastic dominance is necessary but not sufficient for one bidder to be consistently more aggressive than another.

[5] This result is complementary to one in Hopkins (2007). He obtains a similar result, under quite different assumptions. See Section 6.

using the method described above. This section contains the main results, and shows how the most significant existing results follow as simple corollaries of these. Section 4 details how many, if not most, of the remaining existing results can be proven differently, and arguably more easily, using the method proposed here. In Section 5 we focus on the case where distribution functions cross, leading to new results. It is shown how the new theoretical results lead to precise predictions on bidding strategies in the class of examples that are currently used in the numerical literature. In Section 6 we discuss the consequences of letting the support of valuations differ among bidders and describe how the example of Kaplan and Zamir (2007) as well as the result in Hopkins (2007) fit in. Section 7 concludes.

2 Model and preliminaries

We consider a first price auction with n risk neutral and potentially asymmetric bidders with independent private values. Bidder i draws his valuation, v, from an atom-less distribution function, F_i, on $v \in [0, \overline{v}]$, $i = 1, ..., n$. The density, f_i, is assumed to be continuous, as well as finite and strictly positive on $(0, \overline{v}]$. μ_i denotes the expected value of bidder i's valuation. The assumption of a common support is made primarily for expositional simplicity; it is relaxed in Section 6.

We let $b_i(v)$ denote bidder i's equilibrium bidding strategy, and we let $q_i(v)$ denote the equilibrium probability that bidder i with valuation v will win the auction when following his equilibrium strategy.[6]

In this environment, Lebrun (1999) and Maskin and Riley (2000b) have shown that an equilibrium exists, and that bidding strategies are continuous and strictly increasing in valuations.[7] Moreover, bidding strategies are differentiable on $(0, \overline{v}]$. Finally, Lebrun (1999) has shown that $b_i(0) = 0$, implying that $q_i(0) = 0$, for all i, $i = 1, 2, ..., n$. Since $b_i(v)$ is increasing and differentiable on $(0, \overline{v}]$, so is $q_i(v)$. We will take these relatively intuitive properties as given, and examine other properties of the first price auction.

We devote the remainder of this section to some preliminary results concerning the relationships between a given bidder's payoff, his bid, and the probability that he will win the auction. We also discuss what inferences can be made from how

[6] Hence, $q_i(v)$ depends on the equilibrium bid of bidder i and the equilibrium bidding strategies of bidder i's rivals.

[7] Lebrun (1999) also addresses uniqueness, as do Maskin and Riley (2003). If there is a reserve price, the equilibrium is unique. The same holds in a second price auction, as shown by Blume and Heidhues (2004). Lebrun (2006) establishes uniqueness under the condition that F_i is log-concave at $v = 0$ for all i, $i = 1, 2..., n$.

the bidder's payoff compare to the payoff of a rival. In Sections 3 through 6 the interaction of different bidders is then examined.

In the following, bidder i is said to be consistently better off than bidder j if his expected payoff is higher for all interior valuations, $EU_i(v) > EU_j(v)$ for all $v \in (0, \overline{v})$. Similarly, bidder i is consistently more aggressive if he submits higher bids for all interior valuations, $b_i(v) > b_j(v)$ for all $v \in (0, \overline{v})$.

2.1 Payoffs, bids, and winning probabilities

There are two ways to derive expected payoff to a given bidder, both of which are useful. It is straightforward to see that, in equilibrium, bidder i with valuation v has expected payoff of

$$EU_i(v) = (v - b_i(v))\, q_i(v), \tag{1}$$

since his payoff is $v - b_i(v)$ if he wins, which occurs with probability $q_i(v)$.

However, inspired by Myerson's (1981) mechanism design method, it is useful to approach expected payoff from a different angle. If bidder i with valuation v chooses to bid $b_i(z)$, his payoff will be $(v - b_i(z))q_i(z)$. Since he submits the bid which maximizes his payoff, we can write

$$EU_i(v) = \max_z (v - b_i(z))q_i(z).$$

As bidder i with valuation v bids $b_i(v)$ in equilibrium, the problem is maximized at $z = v$.[8] This holds for arbitrary v, and we can now use the Envelope Theorem (while considering v to be a parameter), to conclude that

$$EU_i'(v) = q_i(v). \tag{2}$$

It follows that expected payoff can also be expressed as

$$EU_i(v) = \int_0^v q_i(x)dx.^9 \tag{3}$$

We will use (2) and (3) extensively in the following.

By comparing (1) and (3), the link between winning probabilities and bids becomes apparent,

$$b_i(v) = v - \int_0^v \frac{q_i(x)}{q_i(v)} dx. \tag{4}$$

[8] That is, in equilibrium the buyer submits the bid he is "supposed" to submit given his valuation. Deviating to another bid, or behaving as if his valuation was different, is not profitable.

[9] (2) and thus (3) can also be established by methods that do not require differentiability of $q_i(v)$. Notice that the constant of integration is zero, as a buyer with valuation 0 has payoff 0.

Clearly, bidder i with valuation v bids below his valuation, a phenomenon usually referred to as bid shading.

Lebrun (1999) has also shown that the bid submitted by a bidder with valuation $\overline{v}$ is the same for all bidders, which has important implications for the following analysis.[10] Let $\overline{b}$ denote this common maximal bid. Our first result then follows immediately from (3).

Proposition 1 *In a first price auction, bidder i will never win consistently more often than bidder j, $i \neq j$, regardless of F_i and F_j. That is, it is not possible for $q_i(v) > q_j(v)$ for all $v \in (0, \overline{v})$.*

Proof. Since the highest bid is the same among all bidders, it follows that $q_i(\overline{v}) = 1$ for all i, and that bidders with valuation $\overline{v}$ are equally well off, or, for $j \neq i$,

$$\int_0^{\overline{v}} q_i(x)dx = EU_i(\overline{v}) = \overline{v} - \overline{b} = EU_j(\overline{v}) = \int_0^{\overline{v}} q_j(x)dx. \tag{5}$$

It follows from (5) that it is not possible for bidder i to win consistently more often than bidder j, or $q_i(v) > q_j(v)$ for all $v \in (0, \overline{v})$. If this was the case, the term on the far left in (5) would strictly exceed that on the far right, giving rise to a contradiction. Hence, q_i and q_j either coincide, as is the case with symmetric bidders, or cross at least once.[11] ■

2.2 Ranking payoffs and bounding winning probabilities

In the following we will examine the inferences that can be made by comparing the payoffs of two bidders, bidder i and bidder j. To begin, define $H(b)$ as the distribution function of the highest bid among the $n-2$ rivals to bidder i and bidder j. Thus, $H(b)$ summarizes the relevant information concerning the remaining bidders. Moreover,

[10] See also Maskin and Riley (2003, Lemma 10). Since this plays an important role in the following, we prove it here. By contradiction, assume that $b_1(\overline{v}) \geq b_2(\overline{v}) \geq ... \geq b_n(\overline{v})$, with at least one strict inequality. Then, since it is supposedly optimal for bidder n to bid $b_n(\overline{v})$ when his valuation is $\overline{v}$, we conclude his equilibrium payoff is no smaller than $\overline{v} - b_1(\overline{v})$, the payoff he would get from imitating bidder 1. This coincides with the equilibrium payoff to bidder 1 with valuation $\overline{v}$, and so in turn can be no smaller than what bidder 1 would get from bidding $b_n(\overline{v})$. However, if bidder 1 bids $b_n(\overline{v})$, he earns strictly higher payoff than if bidder n were to bid $b_n(\overline{v})$, i.e. his equilibrium payoff is higher than bidder n's equilibrium payoff. The reason is that a bid of $b_n(\overline{v})$ is sure to beat bidder n, but not bidder 1. Consequently, we have a contradiction.

[11] Lebrun (1999) shows that symmetric buyers must use the same strategy in equilibrium. See also Section 3.

define $q_i^j(v)$ as the probability that bidder i with valuation v outbids bidder j, and define $q_j^i(v)$ analogously. Then, $q_i(v) = H(b_i(v))q_i^j(v)$, i.e. bidder i with valuation v wins if he outbids bidder j, as well as everybody else. Hence, bidder i's equilibrium payoff can be written

$$EU_i(v) = (v - b_i(v))H(b_i(v))q_i^j(v). \tag{6}$$

In equilibrium, bidder i must be at least as well off submitting his equilibrium bid, $b_i(v)$, as the equilibrium bid of bidder j, $b_j(v)$. If he bids $b_j(v)$ he outbids bidder j if bidder j's valuation is below v, which occurs with probability $F_j(v)$. Consequently, it must be the case that

$$EU_i(v) \geq (v - b_j(v))H(b_j(v))F_j(v), \tag{7}$$

and reversing the roles of i and j implies

$$EU_j(v) \geq (v - b_i(v))H(b_i(v))F_i(v). \tag{8}$$

Assume now that there is a valuation v, $v \in (0, \overline{v})$, for which $EU_i(v) > EU_j(v)$. Combining the above results allows the inference that

$$(v - b_i(v))H(b_i(v))q_i^j(v) = EU_i(v) > EU_j(v) \geq (v - b_i(v))H(b_i(v))F_i(v),$$

or, equivalently, that $q_i^j(v) > F_i(v)$. If bidder i is better off than bidder j for all $v \in (0, \overline{v})$, a bound on q_j^i can also be obtained, as demonstrated below. These bounds on the winning probabilities will play a large role in many of the results that follow.

Lemma 1 *If $EU_i(v) > EU_j(v)$ for some $v \in (0, \overline{v})$ then $q_i^j(v) > F_i(v)$. Moreover, if $EU_i(v) > EU_j(v)$ for all $v \in (0, \overline{v})$ then $q_j^i(v) < F_j(v)$ for all $v \in (0, \overline{v})$.*

Proof. The first part was proven above. If $EU_i(v) > EU_j(v)$ for all $v \in (0, \overline{v})$, the first part implies that bidder i with valuation v submits the same bid as bidder j with some valuation z, where $q_i^j(v) = F_j(z) > F_i(v)$ (such a z exists and is unique since bidding strategies are continuous and monotonic and the range of bids is the same for both bidders). From bidder j's point of view, this implies that if he has valuation z, he will bid the same as bidder i with valuation v, and thus outbid bidder i with probability $q_j^i(z) = F_i(v) < F_j(z)$. ■

Lemma 1 is useful for several reason, one of which we explore next. The probability that bidder i outbids bidder j is intimately linked to how the strategies of the two bidders compare. Following Maskin and Riley (2000a), define $p_i(b)$ and $p_j(b)$ as the ex ante probability that bidder i and bidder j, respectively, submits a bid below b. In other words, $p_i(b)$ is the ex ante distribution function of bidder i's bid. Lemma 1 can then alternatively be expressed in terms of these distributions.

Proposition 2 *For any $v \in (0, \overline{v})$, if $EU_i(v) > EU_j(v)$ then $p_i(b) < p_j(b)$ at $b = b_i(v)$ and if $p_i(b) < p_j(b)$ at $b = b_j(v)$ then $EU_i(v) > EU_j(v)$. Moreover, $EU_i(v) > EU_j(v)$ for all $v \in (0, \overline{v})$ if, and only if, $p_i(b) < p_j(b)$ for all $b \in (0, \overline{b})$ (p_i first order stochastically dominates p_j).*

Proof. Since $b_i(v)$ is strictly increasing, $F_i(v)$ coincides with the ex ante probability that bidder i bids below $b_i(v)$. Likewise, $q_i^j(v)$ is the probability that bidder i outbids bidder j with a bid of $b_i(v)$, i.e. it is the ex ante probability that bidder j bids below $b_i(v)$. It follows that if $EU_i(v) > EU_j(v)$ then, by Lemma 1,

$$p_j(b_i(v)) = \Pr(j \text{ bids below } b_i(v)) = q_i^j(v) > F_i(v) = \Pr(i \text{ bids below } b_i(v)) = p_i(b_i(v)). \tag{9}$$

Intuitively, if bidder i is better off than bidder j in equilibrium, it must hold that bidder i wins more often by bidding $b_i(v)$ than bidder j would have by imitating bidder i and also submitting $b_i(v)$. If $EU_i(v) > EU_j(v)$ for all $v \in (0, \overline{v})$ then (9) holds for any v, or, equivalently, for any $b \in (0, \overline{b})$.

The relationship between bid distributions and payoffs goes in the other direction as well. Specifically, if $p_j(b_j(v)) > p_i(b_j(v))$ bidder i could ensure himself a payoff higher than what bidder j experiences by imitating bidder j with a bid of $b_j(v)$. Hence, bidder i with valuation v is strictly better off than bidder j with valuation v. Of course, if this holds for all $b \in (0, \overline{b})$ then bidder i is consistently better off, i.e. better off for all valuations. In this case, the distribution of bids bidder j faces first order stochastically dominates the one bidder i faces, which confirms bidder j must be worse off. ■

Proposition 2 illustrates one of the differences between the approach in this paper and the standard approach. We examine payoffs and use this to inform us about bid distributions. The standard approach starts by examining bid distributions directly. This could then subsequently be used to make inferences concerning payoffs, but this is usually omitted.

Consider now the special case with exactly two bidders in the auction. If bidder i is consistently better off than bidder j, it must necessarily be the case that bidder i is "stronger in expectation", i.e. $\mu_i > \mu_j$.

Lemma 2 *If $n = 2$ and $EU_i(v) > EU_j(v)$ for all $v \in (0, \overline{v})$ then $\mu_i > \mu_j$.*

Proof. If $n = 2$ then $q_i(v) = q_i^j(v)$. If $EU_i(v) > EU_j(v)$ for all $v \in (0, \overline{v})$ Lemma 1 in conjunction with (5) would imply the following relationship between payoff to bidders with valuation $\overline{v}$,

$$\int_0^{\overline{v}} F_i(x)dx < \int_0^{\overline{v}} q_i(x)dx = EU_i(\overline{v}) = EU_j(\overline{v}) = \int_0^{\overline{v}} q_j(x)dx < \int_0^{\overline{v}} F_j(x)dx. \tag{10}$$

As is well-known, integration by parts reveals that

$$\int_0^{\overline{v}} F_i(x)dx = \overline{v} - \mu_i.$$

Consequently, if bidder i is consistently better off than bidder j it necessitates that bidder i has a higher expected valuation than bidder j, $\mu_i > \mu_j$. Otherwise, (10) would produce a contradiction. ■

3 Comparing bidders: Beliefs, payoffs, bids

As in Section 2.2 we focus on a pairwise comparison of bidders. In Section 2.2 we explored some of the inferences that can be made *if* it is known that bidder i is better off than bidder j. We now investigate under what conditions bidder i is better off than bidder j. The approach we suggest also enables us to establish conditions under which bidder i bids more aggressively than bidder j.

Perhaps unsurprisingly, the beliefs bidders have about each other, summarized by the distribution functions, play a dominant role in determining who is better off and who is more aggressive. One possible way of comparing bidder i's beliefs about his rival with bidder j's beliefs about *his* rival would be to consider the ratio

$$F_{i,j}(v) \equiv \frac{F_j(v)}{F_i(v)}, v \in (0, \overline{v}]. \tag{11}$$

Hence, $F_{i,j}$ measures the *relative strength* of bidder i compared to bidder j. For example, if F_i first order stochastically dominates F_j, or $F_i(v) < F_j(v)$ for all $v \in (0, \overline{v})$, the ratio is strictly above 1 in the interior. In this case bidder i is stronger because he is more likely to have a high valuation.

First order stochastic dominance is assumed in virtually the entire existing theoretical literature on asymmetric first price auctions. Indeed, the stronger assumption that the ratio is strictly decreasing is often imposed, or that

$$\frac{f_i(v)}{F_i(v)} > \frac{f_j(v)}{F_j(v)} \text{ for all } v \in (0, \overline{v}], \tag{12}$$

meaning that F_i dominates F_j in terms of the reverse hazard rate. When $F_{i,j}$ is decreasing, bidder i's relative strength is diminishing with the valuation. In other words, the slope of $F_{i,j}$ measures how the relative strength is *changing.*

One of the advantages of the approach presented below is that it makes it (formally and visually) obvious why these assumptions drive the existing results, and why the stronger assumption is often imposed. In addition, it becomes straightforward to derive results for cases where $F_{i,j}$ is not as well-behaved as under the standard assumptions. The number of peaks of the function and the number of times it equals 1 are important factors in predicting how payoffs and bids relate.

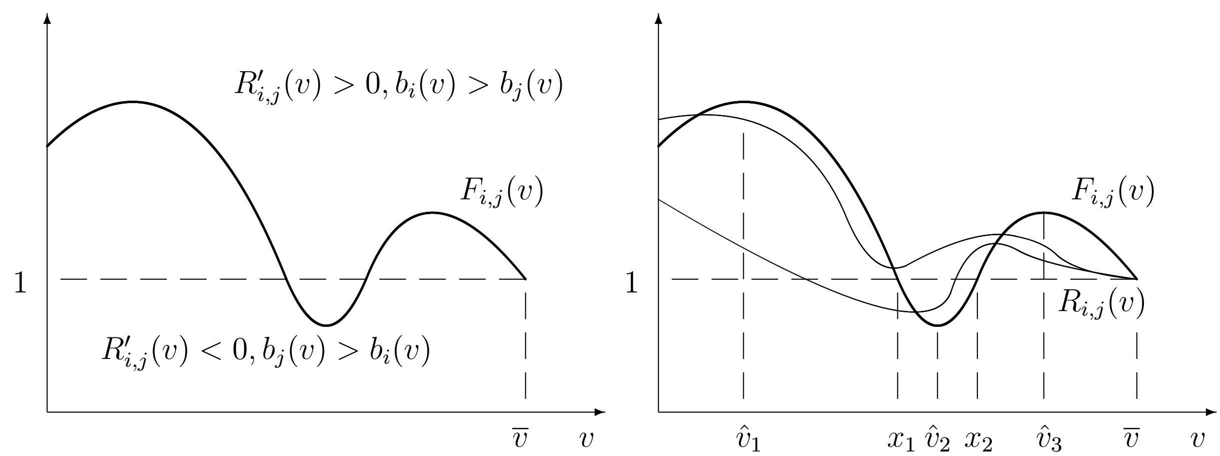
Turning to a comparison of bidder i's payoff with that of bidder j we define

$$R_{i,j}(v) \equiv \frac{EU_i(v)}{EU_j(v)} = \frac{(v - b_i(v))H(b_i(v))q_i^j(v)}{(v - b_j(v))H(b_j(v))q_j^i(v)}, \tag{13}$$

as the ratio of payoffs, where $v \in (0, \overline{v}]$. It is useful to remember that the bidders are equally well off at $\overline{v}$, or $R_{i,j}(\overline{v}) = 1$. At $\overline{v}$, $F_{i,j}$ is also 1, so the two functions coincide at $\overline{v}$.

Lemma 3 and Figure 1 (a) show how $R_{i,j}$ and $F_{i,j}$ compare elsewhere. The important point is that if $R_{i,j}$ is above $F_{i,j}$ then $R_{i,j}$ is increasing. If $R_{i,j}$ is below $F_{i,j}$ then $R_{i,j}$ is decreasing. Given that $F_{i,j}$ is fixed, these properties can then be used to infer how $R_{i,j}$ must behave. Figure 1 (b) describes two possible paths that $R_{i,j}$ may take; they both have the attributes just mentioned.

Incidentally, it is also the case that if $R_{i,j}$ is above $F_{i,j}$ then bidder i is more aggressive than bidder j, $b_i > b_j$.



(a) Comparing $R_{i,j}$ and $F_{i,j}$ (b) Two possible paths for $R_{i,j}$

Figure 1: Comparing $R_{i,j}$ and $F_{i,j}$, and two paths consistent with Lemma 3.

Lemma 3 *For any* $v \in (0, \overline{v}]$, $R_{i,j}(v) >,=,< F_{i,j}(v) \iff b_i(v) >,=,< b_j(v) \iff R'_{i,j}(v) >,=,< 0$.

Proof. If bidder i and bidder j bid the same in equilibrium, $b_i(v) = b_j(v)$, then $q_i^j(v) = F_j(v)$ and $q_j^i(v) = F_i(v)$. In this case, $R_{i,j}(v) = F_{i,j}(v)$, by (13). On the other hand, if bidder i is more aggressive than bidder j, $b_i(v) > b_j(v)$, the observation that $q_j^i(v) < F_i(v)$ combines with (7) to yield the conclusion

$$R_{i,j}(v) = \frac{(v - b_i(v))H(b_i(v))q_i^j(v)}{(v - b_j(v))H(b_j(v))q_j^i(v)} > \frac{(v - b_j(v))H(b_j(v))F_j(v)}{(v - b_j(v))H(b_j(v))F_i(v)} = \frac{F_j(v)}{F_i(v)},$$

or $R_{i,j}(v) > F_{i,j}(v)$. The inequality is reversed if bidder i is less aggressive than bidder j, $b_i(v) < b_j(v)$, since in this case (8) and $q_i^j(v) < F_j(v)$ produce the result. In summary, $R_{i,j}(v) > F_{i,j}(v)$ *if, and only if,* $b_i(v) > b_j(v)$. Moreover,

$$R'_{i,j}(v) = \frac{EU'_i(v)EU_j(v) - EU'_j(v)EU_i(v)}{EU_j(v)^2} = \frac{q_i(v)q_j(v)\left[b_i(v) - b_j(v)\right]}{EU_j(v)^2}, \tag{14}$$

where the last equality follows from (1) and (2). Hence, $R'_{i,j}(v) > 0$ *if, and only if,* $b_i(v) > b_j(v)$. Combined with the previous result the implication is that $R_{i,j}$ *is strictly increasing if it is above* $F_{i,j}$, and strictly decreasing if it is below $F_{i,j}$. At any point where $R_{i,j}$ coincides with $F_{i,j}$ the slope of the former must be zero. ■

Depending on the shape of $F_{i,j}$, Lemma 3 will permit $R_{i,j}$ only a limited number of "paths" through $(0, \overline{v}]$, or, in other words, $R_{i,j}$ can take only a certain number of shapes or forms to be consistent with Lemma 3. This, in turn, allows predictions on the relative payoffs and bids of the two bidders, from which further inferences can be drawn, as demonstrated in Section 2.2.

For example, consider the special case in which the two bidders are symmetric, $F_i = F_j$. Then, $F_{i,j}(v) = 1$ for all $v \in (0, \overline{v}]$. If bidder i, say, bids more aggressively than bidder j for some valuation v, then $R_{i,j}(v) > F_{i,j}(v)$ by Lemma 3. Furthermore, $R'_{i,j}(v) > 0$, meaning that as we move to the right $R_{i,j}$ remains above $F_{i,j}$ and is ever increasing. However, this is impossible since we require that $R_{i,j}$ and $F_{i,j}$ terminate at the same point, $R_{i,j}(\overline{v}) = 1 = F_{i,j}(\overline{v})$. Thus, symmetric bidders must be equally aggressive; they must use symmetric strategies. See Lebrun (1999) for an alternative proof.

The remainder of this section is dedicated to establishing general results concerning payoffs and bids when bidders are asymmetric. In Sections 4 and 5 particular types of asymmetry are examined.

For expositional simplicity, it is assumed that any stationary point of $F_{i,j}$ (should one or more exist) is a strict local maximum or minimum. That is, there are no saddle points, and no intervals over which $F_{i,j}$ is flat. We let m denote the number of stationary points on $(0, \overline{v})$. Notice that m equals the number of times the derivative of $F_{i,j}$ changes sign, i.e. it is the number of intervals on which the derivative has the same sign, less one. If $m > 0$, we let $\widehat{v}_1, \widehat{v}_2, ..., \widehat{v}_m$ list the interior valuations, from smallest to largest, where $F_{i,j}$ is locally maximized or minimized. Letting $\widehat{v}_0 = 0$ and $\widehat{v}_{m+1} = \overline{v}$, we can then partition the interval $(0, \overline{v}]$ into $m + 1$ subsets,

$$(0, \overline{v}] = \cup_{k=0}^{m} (\widehat{v}_k, \widehat{v}_{k+1}] = (0, \widehat{v}_1] \cup (\widehat{v}_1, \widehat{v}_2] \cup ... \cup (\widehat{v}_m, \overline{v}], \tag{15}$$

such that $F_{i,j}$ is monotonic on each subset. In other words, *the relative strength of bidder i is increasing or decreasing on each subset.*

Without loss of generality, we arrange the two bidders such that $F_{i,j}$ approaches one from above as v approaches $\overline{v}$, as in Figure 1. That is, bidder i is the bidder who is "stronger near the top", meaning that he is more likely to have a high valuation as $1 - F_i(v) > 1 - F_j(v)$ if v is close to $\overline{v}$, or $f_i(\overline{v}) > f_j(\overline{v})$.

Let c denote the number of times $F_{i,j}$ equals 1 on $(0, \overline{v})$, and note that c equals the number of intervals over which $F_{i,j}$ is either above or below one, less one. If $c > 0$ we let $x_1, x_2, ..., x_c$ list the valuations, from smallest to largest, where $F_{i,j}$ crosses 1, and let $x_0 = 0$ and $x_{c+1} = \overline{v}$. We will refer to bidder i as locally strong and bidder j as locally weak at v whenever $F_{i,j}(v) > 1$. If $F_{i,j}(v) < 1$ bidder i is locally weak and bidder j locally strong. We can now partition the interval $(0, \overline{v}]$ into $c + 1$ subsets,

$$(0, \overline{v}] = \cup_{k=0}^{c} (x_k, x_{k+1}] = (0, x_1] \cup (x_1, x_2] \cup ... \cup (x_c, \overline{v}], \tag{16}$$

such that $F_{i,j}$ is either above or below one on each subset. Hence, *bidder i is locally stronger or locally weaker than bidder j on alternating subsets.*

Together, m and c summarize the key properties of the competitive environment as described by $F_{i,j}$. In Figure 1, $m = 3$ and $c = 2$. If F_i first order stochastically dominates F_j then $c = 0$. If, moreover, F_i dominates F_j in terms of the reverse hazard rate then $m = 0$ as well. In this paper, we generalize to allow m and c to take any finite number, and show that existing results generalize in natural directions. It is always the case that $m \geq c$.

3.1 Comparing payoffs

By definition, bidder i with valuation v is as well off as bidder j with valuation v whenever $R_{i,j}(v) = 1$. Knowing that $R_{i,j}(\overline{v}) = 1$, we ask how many times $R_{i,j}(v)$ can cross 1 on $(0, \overline{v})$, i.e. how many times bidder i can switch from being better off than bidder j to worse off. The answer follows straightforwardly from Lemma 3.

Theorem 1 *$EU_i(v) = EU_j(v)$ or $R_{i,j}(v) = 1$ no more than c times on $(0, \overline{v})$. More concretely, $R_{i,j}(v) = 1$ at most once on each interval of the form $(x_k, x_{k+1}]$ where* *bidder i is locally strong or weak. Moreover, $R_{i,j}(v) > 1$ or $EU_i(v) > EU_j(v)$ for all $v \in (x_c, \overline{v})$.*

Proof. $R_{i,j}$ can cross one at most once on each interval of the form $(x_k, x_{k+1}]$. Consider, for instance, the interval $(0, x_1)$ in Figure 1, where $F_{i,j}$ is above one (because c is even). If $R_{i,j}(v) = 1$ somewhere on $(0, x_1)$, $R_{i,j}(v)$ must be below $F_{i,j}(v)$ and it follows that $R'_{i,j}(v) < 0$, by Lemma 3. Hence, $R_{i,j}(v) = 1$ for at most one $v \in (0, x_1)$. Indeed, if $R_{i,j}(v) = 1$ for some $v \in (0, x_1)$ it must be the case that $R_{i,j}$ is strictly decreasing and below $F_{i,j}$ on (v, x_1), implying that $R_{i,j}$ cannot cross one at x_1. Conversely, should $R_{i,j}$ equal 1 at x_1 it cannot have crossed 1 on $(0, x_1)$. Hence, there is at most one crossing on $(0, x_1) \cup \{x_1\} = (0, x_1]$. Repeating the argument, there is at most one crossing on $(x_1, x_2) \cup \{x_2\} = (x_1, x_2]$, and so on. It follows that there are at most c crossings on $(0, x_c]$. Regarding the last interval, $(x_c, \overline{v}]$, we already know that $R_{i,j}(\overline{v}) = 1$, which implies that $R_{i,j}$ cannot equal one on $(x_c, \overline{v})$ as well. It follows that $R_{i,j}(v)$ converges to 1 from above as v goes to $\overline{v}$. ■

We can add more details to Theorem 1 by observing that if $R_{i,j} = 1$ somewhere in the interior of an interval where bidder i is locally strong ($F_{i,j} > 1$) then $R_{i,j}$ would cross 1 from above. That is, the strong bidder would be better off than the weak bidder at the beginning of the interval, where his relative strength is increasing, and worse off at the end of the interval, where his relative strength is diminishing.

Notice that if $c = 0$ (first order stochastic dominance), then $R_{i,j}(v) > 1$ for all $(0, \overline{v})$.[12] Given the emphasis on winning probabilities in Section 2, Proposition 3 reiterates and reformulates this result. Notice that q_i can be likened to a distribution function, as it ranges from 0 to 1, and is increasing.

Proposition 3 *Assume $c = 0$ (first order stochastic dominance). Bidder i is consistently better off than bidder j, or $EU_i(v) > EU_j(v)$ for all $v \in (0, \overline{v})$. In other words, q_i is a mean preserving spread over q_j. That is,*

$$\int_0^v q_i(x)dx > \int_0^v q_j(x)dx \tag{17}$$

for all $v \in (0, \overline{v})$, with equality at the endpoints.

(17) implies that $q_i(v) < q_j(v)$ when v is sufficiently close to $\overline{v}$, i.e. the strong bidder wins less often contingent on a high valuation. Intuitively, it is relatively less likely that the weak bidder has a high valuation, and so it is profitable for the strong

[12] If, in addition, $m = 0$ (reverse hazard rate dominance) then $R_{i,j}(v)$ is strictly decreasing on $v \in (0, \overline{v})$ and bounded between 1 and $F_{i,j}(v)$. See Theorem 2 in the next section.

bidder to shade his bid more if his valuation is high. Hence, in the unlikely event that the weak bidder has a high valuation, he is more likely to win.

When $c = 0$, Propositions 2 and 3 imply that the distribution of bidder i's bid first order stochastically dominates the distribution of bidder j's bid. This is arguably one of the most significant results of the existing literature.[13,14]

Corollary 1 *If $c = 0$ (F_i first order stochastically dominates F_j) then $p_i(b) < p_j(b)$ for all $b \in (0, \bar{b})$ (p_i first order stochastically dominates p_j).*

Turning to the possibility that $c > 0$, the last part of Theorem 1 means that the bidder who is strong near the top, bidder i, is also better off near the top. However, it is possible that bidder i is not consistently better off.

As an example with $c > 0$, Figure 1 (b) replicates Figure 1 (a) but overlays it with two qualitatively different possible paths for $R_{i,j}$ that is consistent with Lemma 3 (there are other possible paths). As implied by Theorem 1, $R_{i,j}$ crosses 1 zero, one or two times on the interior (the two paths in Figure 1 (b) illustrate the extreme cases with no or two crossings).

While Theorem 1 gives only an upper bound on the number of crossings, it is possible in some circumstances to narrow down the number of possibilities substantially. For example, if $n = 2$ and $\mu_i \leq \mu_j$ then $R_{i,j}$ must cross 1 at least once on $(0, \bar{v})$, by Lemma 2, thus ruling out one of the paths depicted in Figure 1 (b). Recall the important implication that when $R_{i,j}$ crosses 1, the bid distributions, p_i and p_j, must also cross (Proposition 2).

An alternative approach would be to attempt to pin down the properties of $R_{i,j}$ as v converges to 0. Assuming densities are finite and strictly positive at $v = 0$, Fibich et al (2002) study the first order conditions as $v \to 0$. As mentioned earlier, it is known that bidding strategies are differentiable on $(0, \bar{v}]$. However, it does not appear to have been proven that the limit of the derivative exists as $v \to 0$.

[13] As mentioned, Proposition 2 goes both ways. Hence, from the known result that $p_i(b) < p_j(b)$ for all $b \in (0, \bar{b})$ if $F_i(v) < F_j(v)$ for all $v \in (0, \bar{v})$, it is possible to conclude that bidder i is consistently better off than bidder j (though this does not appear to have been explicitly stated before). However, any result on how payoffs and bid distributions compare for the case of $c > 0$ is new. See also Section 5.

[14] Although winning probabilities as a function of valuation intersect, the strong bidder is still more likely, ex ante, than the weak bidder to win the auction. To see this, start with the following thought experiment. Rather than drawing a valuation from the distribution function (and subsequently calculating the bid), we can think of a bidder as drawing a bid from the distribution of bids (and later inferring the valuation, if necessary). This, of course, leads to the insight that the strong bidder is more likely to "draw", or submit, a high bid, and it follows that the strong bidder is more likely than the weak bidder to win the auction ex ante.

Nevertheless, Fibich et al (2002) use L'Hospital's rule to investigate the properties of the derivative of the bidding strategy in the limit. The appropriate interpretation of their result therefore seems to be that *if* the limit of the derivative exists as $v \to 0$, *then* this limit is the same for all bidders. This, in turn, would imply that bidding strategies are virtually identical when v is small.[15] In this case, it is easy to show that $R_{i,j}$ and $F_{i,j}$ would converge to the same number as $v \to 0$. In the example in Figure 1, we would therefore conclude that $R_{i,j}(v) = 1$ either zero or two times in the interior. If it is also the case that $n = 2$ and $\mu_i \leq \mu_j$, it must therefore be the case that $R_{i,j}(v) = 1$ twice in the interior, meaning that bidder i is better off than bidder j for low and high valuations, but worse off for intermediate valuations.

There are two reason to be careful about relying too much on the arguments in the previous paragraph. The first is the use of L'Hospital's rule. The second is the fact that there are many compelling examples in which densities are not strictly positive or finite at $v = 0$. For example, if a set of potential bidders collude and bidder i is the member with the highest valuation, bidder j would perceive the density of the cartel representative to be zero even if the densities of the individual cartel members are strictly positive.[16]

Moreover, many commonly used distributions do not have strictly positive or finite densities at $v = 0$. Examples include the beta distribution, the Weibull distribution, the log-normal distribution and the commonly used power distribution. Plum (1992) and Cheng (2006) analytically derive bidding strategies in different classes of situations where bidders draw valuations from power distributions (over different supports). It is also interesting to note that a significant portion of the numerical work on asymmetric auctions focusses on these types of examples. For instance, Marshall et al (1994) consider power distributions. Gayle and Richard (2005) consider Weibull distributions and log-normal distributions, although the latter are truncated away from zero.

3.2 Comparing bids

Lemma 3 shows that the bids of bidder i and bidder j can be compared by comparing $R_{i,j}$ with $F_{i,j}$. Whenever the two curves coincide, the bids coincide as well, and vice versa. Following the approach in the previous subsection, we thus seek to determine the maximal number of crossings between $R_{i,j}$ and $F_{i,j}$. In this case the number of

[15] See Lebrun (2006, footnotes 2 and 8) for a discussion of the problems with this approach, and, in particular, the use of L'Hospital's rule.

[16] If buyer 1 and 2 collude, the distribution of the highest valuation among these two bidders is $F_1(v)F_2(v)$, the density of which is zero at $v = 0$ if $f_1(0)$ and $f_2(0)$ are finite.

peaks of $F_{i,j}$, m, emerges as an upper bound. Moreover, a lower bound is easily added when $c > 0$.

Theorem 2 *$b_i(v) = b_j(v)$ or $R_{i,j}(v) = F_{i,j}(v)$ no more than m times on $(0, \overline{v})$. More concretely, $R_{i,j}(v) = F_{i,j}(v)$ at most once on each interval of the form $(\widehat{v}_k, \widehat{v}_{k+1}]$ where bidder i's relative strength is increasing or decreasing. Moreover, $R_{i,j}(v) < F_{i,j}(v)$ or $b_i(v) < b_j(v)$ for all $v \in (\widehat{v}_m, \overline{v})$.*[17] *Finally, if $c > 0$ there exists at least one valuation, $v \in (0, \overline{v})$, for which $R_{i,j}(v) = F_{i,j}(v)$.*

Proof. $R_{i,j}$ can cross $F_{i,j}$ at most once on each interval of the form $(\widehat{v}_k, \widehat{v}_{k+1})$, because at any point of crossing $R_{i,j}$ would be flat, by Lemma 3, whereas $F_{i,j}$ is strictly monotonic with a non-zero derivative on the interval. Replicating the argument from the proof of Theorem 1 then establishes that there can be at most one crossing on $(\widehat{v}_k, \widehat{v}_{k+1}]$, and thus at most m crossings on $(0, \widehat{v}_m]$. Moreover, since $R_{i,j}(\overline{v}) = F_{i,j}(\overline{v})$ there can be no crossing on $(\widehat{v}_m, \overline{v})$. It must then be the case that $R_{i,j}(v) < F_{i,j}(v)$ for $v \in (\widehat{v}_m, \overline{v})$, because otherwise $R_{i,j}$ would be increasing and diverge from $F_{i,j}$. Finally, when $c > 0$ or $F_{i,j}(v) = 1$ for some $v \in (0, \overline{v})$, $R_{i,j}$ must necessarily cross $F_{i,j}$ as we move left from $\overline{v}$. The reason is that $R_{i,j}(\overline{v}) = 1$ and that $R_{i,j}$ is decreasing below $F_{i,j}$ (in Figure 1 any path that $R_{i,j}$ can take crosses $F_{i,j}$ between x_2 and $\widehat{v}_3$). ■

Notice that if bidding strategies cross on an interval where bidder i is becoming increasingly strong, then $R_{i,j}$ crosses $F_{i,j}$ from above. That is, bidder i would be more aggressive than bidder j on the first portion of the interval, where his relative strength is the smallest, and less aggressive towards the end, where his relative strength is largest.

In summary, the bidding strategies of bidder i and bidder j cross at most m times.[18] If $m = 0$ (reverse hazard rate dominance) then $R_{i,j}(v) < F_{i,j}(v)$ for all $v \in (0, \overline{v})$, meaning that bidder j, who is weaker everywhere, bids more aggressively than bidder i. Alongside Corollary 1, this is among the primary results of the existing literature.

Corollary 2 (Sufficient Condition) *If $m = 0$ (reverse hazard rate dominance) then bidder j is consistently more aggressive than bidder i, i.e. $b_j(v) > b_i(v)$ for all $v \in (0, \overline{v})$.*

[17] Hence, bidder j, who is weaker at the top, bids more aggressively than bidder i for high valuations. This result was first established by Fibich et al (2002), who proved it by examining the system of differential equations. We add the observation that the bidder who is weak at the top is also worse off at the top, by the last part of Theorem 1.

[18] The standard approach can also be used to prove this fact, i.e. that bidding strategies coincide at most once on any interval where $F_{i,j}$ is strictly monotonic. However, many results, such as Corollary 3 and Proposition 4 below, are more easily proven with the current approach.

The last part of Theorem 2 has an important implication which is worth stating explicitly. Specifically, we see that first order stochastic dominance ($c = 0$) is *necessary* for one bidder to be consistently more aggressive than another. This insight appears to be new to the literature.

Corollary 3 (Necessary Condition) *If bidder j is consistently more aggressive than bidder i then $c = 0$, i.e. $F_i(v) < F_j(v)$ for all $v \in (0, \overline{v})$.*

Corollary 3 means that first order stochastic dominance is necessary for the weak bidder to be consistently more aggressive, but Corollary 2 suggests that it may not be sufficient. The by now familiar arguments prove that if $F_{i,j} \to 1$ as $v \to 0$, then $R_{i,j}$ must cross $F_{i,j}$ at least once. For example, if $F_i < F_j$ in the interior (first order stochastic dominance) but the densities are finite and strictly positive and coincide at $v = 0$, i.e. $f_i(0) = f_j(0) \in (0, \infty)$, then bidder i, the strong bidder, must be *more aggressive* than the weak bidder for a set of types close to zero.[19] This establishes a *whole class of situations* in which it is *not true* that the weak bidder is always more aggressive than the strong bidder.[20]

When $m > 0$ it is in specific cases often possible to make precise the number of times the bid functions will cross. In particular, consider the possibility that $F_{i,j}$ *oscillates* around 1, in the sense that the peaks are alternatingly below and above 1, and assume moreover that these peaks become less pronounced as we move to the right. The following definition makes this more precise.

Definition 1 *$F_{i,j}$ has the "diminishing wave property" if*

(i) $1 < F_{i,j}(\widehat{v}_m) < F_{i,j}(\widehat{v}_{m-2}) < F_{i,j}(\widehat{v}_{m-4})...,$

(ii) $1 > F_{i,j}(\widehat{v}_{m-1}) > F_{i,j}(\widehat{v}_{m-3}) > ...,$ *and*

(iii) *$F_{i,j}$ is not both maximized and minimized on the interior (that is, $F_{i,j}$ is maximized or minimized as $v \to 0$).*

[19] By contradiction, if bidder i is consistently less aggressive than bidder j then $F_{i,j}(v) > R_{i,j}(v)$ for any $v \in (0, \overline{v})$, which implies the latter is decreasing. However, since $R_{i,j}(v) > 1$ for any $v \in (0, \overline{v})$, $R_{i,j}$ must necessarily intersect $F_{i,j}$ as we move towards 0.

[20] Assume, for example, that $F_j(v) = v + .4v^2(1 - v^2)$ and $F_i(v) = v - .4v^2(1 - v^2)$, $v \in [0, 1]$, in which case $F_{i,j} > 1$ on the interior with $F_{i,j} \to 1$ as $v \to 0$. Fibich and Gavious (2003) use numerical methods to derive expected revenue in this particular example (see their Table 1), but they do not plot bidding strategies or comment on whether they intersect. Maskin and Riley (2000a, footnote 14) propose another example with the same property, $f_i(0) = f_j(0)$, and point out that bidding strategies must cross in that specific example. See also Section 6.

Figure 2 (a) illustrates the shape we have in mind, while Figure 2 (b) and Figure 2 (c) give concrete examples.[21] Notice that $c = m$ ($F_{i,j}$ always crosses 1 to the left of a peak). In this case, $R_{i,j}$ will cross $F_{i,j}$ exactly m times, i.e. the upper bound found in Theorem 2 is "binding".

Proposition 4 *If $F_{i,j}$ has the diminishing wave property then $R_{i,j}$ crosses $F_{i,j}$ exactly m times on* $(0, \overline{v})$.

Proof. Starting at $\overline{v}$ and moving left, $R_{i,j}$ must cross $F_{i,j}$ on $(x_m, \widehat{v}_m]$, by the argument given in Theorem 2. At this crossing, $R_{i,j}$ is, by assumption, between $F_{i,j}(\widehat{v}_{m-1})$ and $F_{i,j}(\widehat{v}_{m-2})$ in value. Moving further to the left, $R_{i,j}$ is above $F_{i,j}$ and thus increasing. Hence, it must cross $F_{i,j}$ again somewhere on $(\widehat{v}_{m-2}, \widehat{v}_{m-1})$, at which place $R_{i,j}$ will take a value between $F_{i,j}(\widehat{v}_{m-3})$ and $F_{i,j}(\widehat{v}_{m-2})$. As we continue leftward, $R_{i,j}$ must intersect $F_{i,j}$ once on each monotonic segment of $F_{i,j}$. ■

Similar arguments can be applied to the interval $(\widehat{v}_1, \overline{v})$ in the example in Figure 1 since the "waves" are diminishing from $\widehat{v}_1$ onwards. Hence, bidding strategies must cross twice on $(\widehat{v}_1, \overline{v})$, as is depicted in Figure 1 (b).

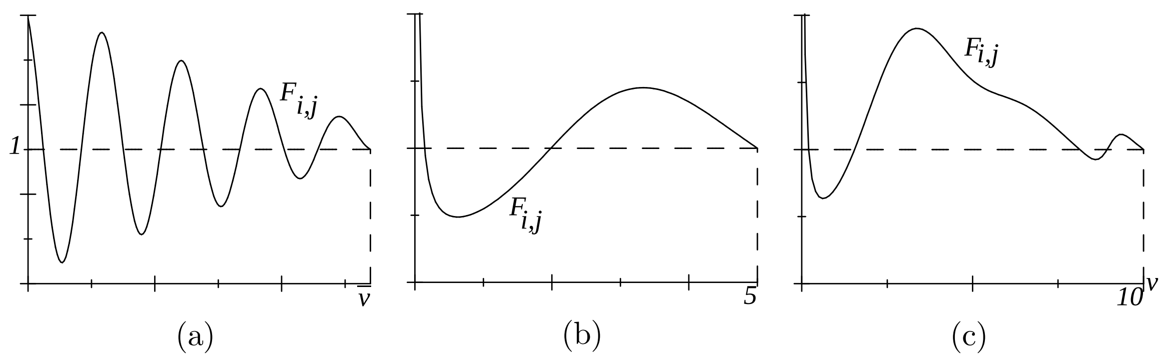


Figure 2: Stylized shape of diminishing waves, and two examples.

[21] In Figure 2(b), $F_i = (v/5)^2$, $v \in [0,5]$, while F_j is a normal distribution with mean 3 and standard deviation 1, truncated on $[0,5]$. In Figure 2 (c), $F_i(v) = (v/10)^2$, $v \in [0,10]$, and $F_j(v) = \frac{1}{3}(G_1(v) + G_2(v) + G_3(v))$ where G_i is the cdf of a normal distribution with mean η_i and standard deviation σ_i, truncated on $[0,10]$, $i = 1,2,3$. Further, $\eta_i = 3i$, while $\sigma_1 = \sigma_2 = 1$ and $\sigma_3 = 0.25$.

4 Strong and weak bidders: Additional results

As mentioned, Corollary 1 and Corollary 2 are arguably the main results of the current literature on asymmetric first price auctions. In this section we prove a number of other known results, all of which assume first order stochastic dominance or reverse hazard rate dominance. Specifically, we consider bidders' preferences for different auction formats, as well as for the type of competition they face. Bids and bid distributions are then compared to their counterparts in symmetric auctions. We posit that the proofs offered here are simpler than earlier proofs.

Assuming there are only two bidders, and maintaining the assumption that (12) holds, or $m = 0$, Maskin and Riley (2000a) show that the strong bidder, bidder $i = 1$, prefers the second price auction to the first price auction, but that the weak bidder, bidder $j = 2$, feels the other way. In fact, this follows easily from Corollary 2. The reason is that it implies that $q_2(v) > F_1(v)$ and $q_1(v) < F_2(v)$ in the first price auction. In contrast, since the second price auction is efficient, bidder 2 with valuation v would win such an auction with probability $F_1(v)$. Hence, the weak bidder wins more often in the first price auction than in the second price auction, while the opposite holds for the strong bidder. The result then follows from (3).[22]

Corollary 4 *If $n = 2$ and $m = 0$ (reverse hazard rate dominance) the strong bidder strictly prefers the second price auction to the first price auction, while the weak bidder strictly prefers the first price auction to the second price auction.*

Maskin and Riley (2000a, footnote 16) claim that the ranking holds for bidders with high valuations even if (12) is replaced with the weaker assumption of first order stochastic dominance, $c = 0$. Lemma 1 can be utilized to prove this.

Corollary 5 *If $n = 2$ and $c = 0$ (first order stochastic dominance) the strong bidder with valuation $\overline{v}$ strictly prefers the second price auction to the first price auction, while the opposite holds for the weak bidder with valuation $\overline{v}$. Moreover, $\overline{b} \in (\mu_2, \mu_1)$.*

Proof. By Proposition 3 and Lemma 1, $q_1(v) > F_1(v)$ which implies

$$EU_2(\overline{v}) = EU_1(\overline{v}) = \int_0^{\overline{v}} q_1(x)dx > \int_0^{\overline{v}} F_1(x)dx,$$

[22] This argument also proves that the result extends to the case where there are more than one weak bidder and one strong bidder (symmetric bidders must use symmetric equilibrium strategies). Likewise, if there are more bidders, and these can be arranged according to (12), the weakest bidder prefers the first price auction, and the strongest bidder the second price auction.

where the term furthest to the right equals the expected payoff to a weak bidder with valuation $\overline{v}$ in a second price auction. Thus, the weak bidder with valuation $\overline{v}$ prefers the second price auction. Since he wins with probability one in either auction, his expected payment in a second price auction, μ_1, must be less than his expected payment in a first price auction, $\overline{b}$. Next, the second part of Lemma 1 implies that

$$EU_1(\overline{v}) = EU_2(\overline{v}) = \int_0^{\overline{v}} q_2(x)dx < \int_0^{\overline{v}} F_2(x)dx,$$

where the last term equals the expected payoff to a strong bidder with valuation $\overline{v}$ in a second price auction. An argument similar to that given before leads to the conclusion that $\overline{b}$ exceeds the expected valuation of the weak bidder, μ_2. ■

As we have seen, the weak bidder may be more aggressive than the strong bidder. This raises the question of whether the strong bidder is better off facing a weak, but aggressive bidder, rather than another strong bidder, with the same distribution function as himself. Despite the aggressiveness, we show next that both a weak and a strong bidder prefer facing a weak rather than a strong bidder. Let F_s and F_w denote two distribution functions, the first of which first order stochastically dominates the latter.

Corollary 6 *Assume $n = 2$, $F_s(v) < F_w(v)$ for all $v \in (0, \overline{v})$, and bidder i draws his valuation from either F_s or F_w. Regardless of whether bidder i is weak or strong himself, he is better off facing a weak rather than a strong rival. That is, bidder i with valuation v, $v \in (0, \overline{v}]$, is strictly better off if bidder j, $j \neq i$, draws his valuation from F_w rather than F_s.*[23]

Proof. First, assume that bidder i is strong, $F_i = F_s$. Then, we have already established that $q_i(v) > F_s(v)$ for all $v \in (0, \overline{v})$ if $F_j = F_w$. On the other hand, $q_i(v) = F_s(v)$ for all $v \in (0, \overline{v})$ if $F_j = F_s$, which follows from the fact that symmetric bidders bid symmetrically. Hence, for all $v \in (0, \overline{v})$, bidder i wins more often in equilibrium if bidder j is weak rather than strong. By (3), he is strictly better off facing a weak bidder.

Next, consider the possibility that bidder i is weak, $F_i = F_w$. To begin, assume bidder j is strong, $F_j = F_s$. In this case, bidder i's winning probability, $q_w(v)$, is less than $F_w(v)$ (Lemma 1). On the other hand, if bidder i were to face another weak bidder he would win with probability $F_w(v)$. Again, the result follows from (3). ■

[23] This result also holds in the second price auction. It can also be extended to the case with $n-1$ bidders of the same kind (weak or strong), who would prefer the remaining bidder to be weak.

This result is related to a result in Lebrun (1998), who shows that if one bidder becomes much stronger, the other bidder is worse off. That is, Lebrun (1998) assumes $m = 0$ rather than $c = 0$.

Maskin and Riley (2000a) also compare bids and bid distributions in the asymmetric auction with their counterparts in a symmetric auction where both bidders draw valuations from the same distribution. The above corollaries will allow us to easily reproduce their results in the following.[24]

Let $\pi_s(b)$ and $\pi_w(b)$ denote the distributions of bids in a symmetric auction where *both* bidders are strong or weak (they both draw from F_s or F_w), respectively. Following the earlier notation, let $p_s(b)$ and $p_w(b)$ denote the distribution of bids in the asymmetric auction where one bidder is strong, the other is weak. The following Corollary proves that these distributions relate as in Figure 3.

Corollary 7 *Assume $n = 2$ and $m = 0$ (reverse hazard rate dominance). Bid distributions can be ranked according to first order stochastic dominance. Specifically, π_s dominates p_s, which dominates p_w, which in turn dominates π_w.*

Proof. Given Corollary 4, in an asymmetric environment the weak bidder is better off in a first price auction than in a second price auction. In a second price auction he would win with probability $F_s(v)$, which coincides with the probability with which he would win in a first price auction if he himself was also strong (and the bidders therefore symmetric). In light of (3), this leads to the conclusion that in a first price auction the bidder is better off, for a fixed valuation, against a strong bidder if that rival believes the first bidder is weak (in which case the winning probability exceeds $F_s(v)$) rather than strong (where the winning probability is $F_s(v)$). In other words, the bidder is better off facing the distribution of bids p_s than π_s (these summarize how the strong rival would react to the different beliefs). This implies that π_s first order stochastically dominates p_s; if it did not, the bidder could ensure himself a higher payoff when facing π_s simply by duplicating his own strategy when facing p_s. A similar thought experiment involving the strong bidder shows that p_w must first order stochastically dominate π_w. It remains to show that p_s dominates p_w. However, this has already been established, in Corollary 1. ■

Given Corollary 7 or Figure 3 it is also possible to rank bidders' bids in the three different environments (two weak, two strong, or one of each).

Corollary 8 *Assume $n = 2$ and $m = 0$. Regardless of whether a bidder is weak or strong himself, he bids more aggressively if facing a strong rather than a weak rival.*

[24] Corollaries 7 and 8 combine Propositions 3.3 and 3.5 in Maskin and Riley (2000a). See also Lebrun (1998).

Proof. Fix a valuation of the weak bidder, say, and locate the corresponding probability, $F_w(v)$, on the vertical axis in Figure 3. Since p_w is further to the right than π_w, we conclude that the weak bidder bids more aggressively if he faces a strong bidder (in which case p_w describes his response) rather than a weak bidder (when π_w describes his response). Reproducing the argument for the strong bidder shows that he responds in the same way. ■

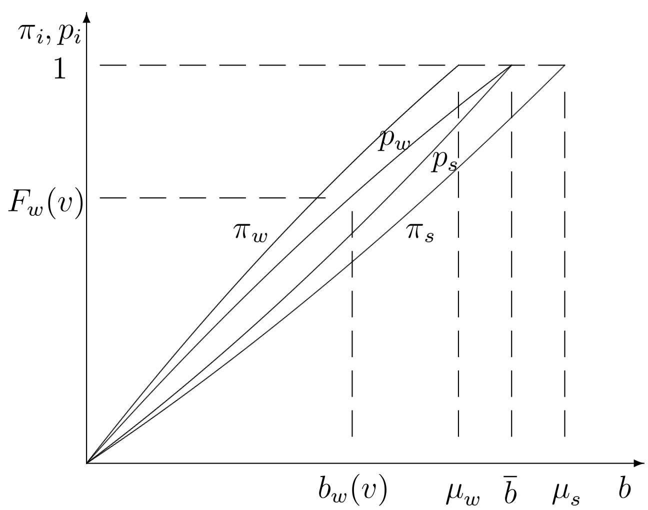


Figure 3: Bid distributions in different competitive environments.

5 Beyond first order stochastic dominance

The standard assumption in theoretical work is that $c = 0$ (first order stochastic dominance), often supplemented with the stronger monotonicity condition that $m = 0$ (reverse hazard rate dominance).[25] In this section we consider situations where $F_{i,j}$ is less well-behaved. We start with a brief discussion of the assumptions in numerical work, where neither monotonicity nor first order stochastic dominance is necessarily assumed. In this regard, our focus is on bidding strategies, though it must be pointed out that numerical methods are also used effectively to compute expected revenue.

[25] As mentioned, Fibich. et al. (2002) study bidding strategies near the top without the assumption of first order stochastic dominance. Kaplan and Zamir (2007) examine bidding strategies when bidders draw valuations from different uniform distributions, in which case first order stochastic dominance may not hold either. Hopkins (2007) does not assume first order stochastic dominance, but his new result requires the support of different bidders to be different in a rather specific way (see Section 6).

In the first paper to use numerical methods, Marshall et al (1994) assume bidders draw valuations from different power distributions. In this case $F_{i,j}$ is particularly well-behaved, as $c = m = 0$. Hence, bidding strategies do not cross. Later papers have examined other types of asymmetry.

More concretely, Bajari (2001) and Li and Riley (2006) assume bidders draw valuations from different truncated *normal distributions*.[26] Gayle and Richard (2005) give examples where bidders draw valuations either from different truncated *Weibull distributions* or different truncated *log-normal distributions*. Gayle and Richard (2005) explicitly point out that bidding strategies may cross. However, it turns out that *all existing classes of numerical examples of asymmetry in first price auctions share a common feature; m is at most one*.[27] Therefore, with the specifications currently being used in numerical work, bidding strategies will be found to cross at most once. As we discuss in Section 6, this result depends on the assumption of a common support, which is also typically used in numerical work.

Proposition 5 *Bidding strategies will cross at most once if bidder i and bidder j draw valuations from different power distributions, normal distributions, Weibull distributions, or log-normal distributions.*

Proof. It is necessary to show only that $m \leq 1$. It is straightforward (and therefore omitted) to show that f_j/f_i has at most one interior peak in all the above examples. As observed by e.g. Hopkins and Kornienko (2007) this in turn implies that $F_{i,j}$ has at most one interior peak. ■

Notice that in all the aforementioned examples, both bidders draw from distributions of the *same type*. However, as Figure 2 (b) and Proposition 4 illustrate, if bidders draw from different types of distributions, e.g. one from a power distribution and the other from a normal distribution, $F_{i,j}$ may have more peaks and bidding strategies may cross more times.

In the following we relax the assumption of first order stochastic dominance, or $c = 0$ (the distribution functions do not cross). We assume there are exactly two bidders, with bidder $i = 1$ being the bidder who is stronger near the top.

We argue that once first order stochastic dominance (FOSD) is well understood the natural next step would be to consider *second order stochastic domi-*

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[26] Li and Riley (2006) also consider uniform distributions (a special case of power distributions). They have developed the freely available program BIDCOMP^2, which allows the user to estimate bidding strategies when bidders draw valuations either from different power distributions or from different normal distributions.

[27] Fibich et al (2002) and Fibich and Gavious (2003) examine a few other examples, but there is little to unify these.

nance (SOSD). Notice that SOSD encompasses FOSD (the latter implies the former). Hence, when we refer to SOSD we need to keep in mind that FOSD may or may not be satisfied. The new and interesting case is when SOSD is satisfied, but FOSD is not ($c > 0$).

By definition, F_2 *second order stochastically dominates* F_1 if

$$\int_0^v F_1(x)dx > \int_0^v F_2(x)dx \text{ for all } v \in (0, \overline{v}). \tag{18}$$

Notice that due to the convention that bidder 1 is strong near the top, it is now F_2 that stochastically dominates F_1. Specifically, F_2 second order stochastically dominates F_1, although F_2 does not first order stochastically dominate F_1 because F_1 dips below F_2 for sufficiently high valuations ($c > 0$).

(18) holds with equality at $\overline{v}$ if and only if $\mu_2 = \mu_1$, in which case F_1 *is a mean preserving spread over* F_2. Hence, this type of asymmetry can be used to describe situations where bidder 1 is "unpredictable" and bidder 2 "predictable"; it is more difficult to "guess" the valuation of bidder 1 than that of bidder 2 even though the expected valuations are identical. For instance, it is easily shown that F_1 has a higher variance than F_2. Likewise, F_1 has more mass at the tails than F_2.

The following is a compelling example of this type of asymmetry: Bidders draw valuations from different normal distributions, truncated on $[0, \overline{v}]$, with the common mean $\mu_1 = \mu_2 = \frac{1}{2}\overline{v}$. The difference is that the distribution from which the unpredictable bidder draws his valuation has larger variance than the distribution from which the predictable bidder draws his valuation, $\sigma_1 > \sigma_2$. In this example, $m = c = 1$, $\mu_1 = \mu_2$, and (18) holds.

If (18) does not hold with equality at $\overline{v}$ then $\mu_2 > \mu_1$. Nevertheless, F_1 has more mass at the tails than F_2, i.e. *bidder 1 is more likely to have valuations that are extreme* (high or low).

Given SOSD holds but FOSD does not, we conclude that bidding strategies must cross, as $c > 0$ (Theorem 2). They cross exactly once if $m = 1$. Moreover, bidder 1 is better off than bidder 2 for high valuations, since he is stronger near the top (Theorem 1). However, since $\mu_2 \geq \mu_1$, bidder 2 is better off for a set of valuations (Lemma 2).

Since neither bidder is consistently better off *bid distributions must cross* (Proposition 2). This is a new result, and should be contrasted with the fact that bid distributions do *not* cross under FOSD (Corollary 1).[28]

With more "regularity assumptions" on the shape of $F_{i,j}$ we can make more precise predictions regarding the relationship between the two bid distributions.

[28]However, Hopkins (2007) contains a related result, which we discuss in Section 6.

Proposition 6 *Assume* $n = 2$ *and that* F_2 *second order stochastically dominates* F_1*, yet first order stochastic dominance does not apply. Then* (*i*) *bid distributions,* p_1 *and* p_2*, must cross on* $(0, \bar{b})$*, and* (*ii*) *if* $c = 1$ *then* p_1 *has more mass at the tails than* p_2*, i.e. the bidder who is more likely to have extreme valuations is also more likely to submit extreme bids. Moreover, if* (*iii*) $m = c = 1$ *then* $p_2(b)/p_1(b)$ *is single peaked and crosses* 1 *exactly once, like* $F_{1,2}$*.*

Proof. The first part follows from Proposition 2. For the second part, notice that there is a unique valuation for which the two bidders are equally well off when $c = 1$ (Theorem 1). Since bidder 1 is better off near the top, Proposition 2 implies that $p_1(b) < p_2(b)$ when b is close to $\bar{b}$. The opposite holds for bids close to zero since bidder 2 is better off near the bottom. To prove the third part, with $m = 1$, it is probably easiest to revert back to the standard approach to asymmetric first price auctions. As in Maskin and Riley (2000a), it can be established that the derivative of $p_2(b)/p_1(b)$ is proportional to $\left(b_1^{-1}(b) - b\right)^{-1} - (b_2^{-1}(b) - b)^{-1}$, which is zero exactly once since bidding strategies coincide exactly once. Hence, the ratio has a unique stationary point; it is single-peaked.[29] Thus, there is a unique bid at which the ratio is one. ■

Given (18), notice that bidder 2 would be *consistently* better off than bidder 1 in a *second price auction*, but this is not the case in a first price auction where bidder 1 is better off near the top. Hence, the auction format influences which bidder is better off. It should be pointed out that this is not the case when comparing strong and weak bidders, where the strong bidder is consistently better off in both auctions.

In auctions with weak and strong bidders it was possible, from the bidders' point of view, to rank the first price auction and the second price auction as well as the type of competition they face.

Here, with predictable and unpredictable bidders, we are able to rank auctions at the bottom, i.e. for low valuations. We assume $m = c = 1$. Then, if v is sufficiently low, the unpredictable bidder, bidder 1, prefers a first price auction to a second price auction, while the opposite holds for the predictable bidder. This follows from the fact that $b_1 > b_2$ when v is small, implying that bidder 1 wins more often in a first price auction than in a second price auction, $q_1(v) > F_2(v)$, and vice versa for bidder 2.

Regarding a ranking for all valuations, at least one of the bidders will consistently prefer one auction to the other, but it is possible that the other bidder's preference

[29] The same argument shows that if $c = m = 0$ (reverse hazard rate dominance), then p_2/p_1 is strictly decreasing. More generally, if $F_{1,2}$ has the "diminishing wave" property in Proposition 4, then p_2/p_1 has exactly m stationary points.

depends on his valuation. Since $b_2 > b_1$ if v is high, or $q_1 < F_2$, the first price auction becomes less attractive relative to the second price auction for bidder 1 as the valuation increases (and vice versa for bidder 2).

The only reason a global ranking is not possible is because it has proven impossible to pinpoint the value of $\overline{b}$ and thereby payoff at the top, $EU_i(\overline{v}) = \overline{v} - \overline{b}$. Recall that in a second price auction bidder i's payoff at the top is $\overline{v} - \mu_j$, $j \neq i$. Hence, to compare payoff at the top we need to compare $\overline{b}$ with μ_1 and μ_2. A natural conjecture is that the last part of Corollary 5 generalizes. In fact, in all known numerical examples it is true that the maximal bid is between the two expected values. In this case, $\mu_2 \geq \overline{b} \geq \mu_1$ would imply that bidder 1 with valuation $\overline{v}$ would weakly prefer the first price auction, while bidder 2 with valuation $\overline{v}$ would weakly prefer the second price auction. This, in turn, would imply that bidder 1 prefers the first price auction for all $v \in (0, \overline{v})$, while bidder 2 would prefer the second price auction for all $v \in (0, \overline{v})$. Likewise, bidders would also prefer facing an unpredictable rather than a predictable bidder. Thus, establishing whether it is generally true, with two bidders, that the maximal bid falls between the expected values would lead to several new auxiliary results. We propose this as future research topic.

Example 1: Gayle and Richard (2005) use numerical methods to examine a situation where bidders draw valuations from the Weibull distributions

$$F_1(v) = \frac{1 - e^{-\left(\frac{v}{1.5}\right)^{0.5}}}{1 - e^{-\left(\frac{4}{1.5}\right)^{0.5}}}, \quad F_2(v) = \frac{1 - e^{-\left(\frac{v}{1.11}\right)^{1.5}}}{1 - e^{-\left(\frac{4}{1.11}\right)^{1.5}}}$$

for $v \in [0, 4]$. In this example it is easily verified that $m = c = 1$ and that the expected values are approximately $\mu_1 \approx 0.84 < 0.998 \approx \mu_2$. Together, these properties imply SOSD and the results discussed above. Gayle and Richard's (2005) simulations confirm our theoretical result that bidding strategies will cross exactly once. However, they calculate neither payoff nor bid distributions, so in this case the theoretical results are more plentiful than the numerical results. It is, however, apparent from their analysis that $\overline{b} \in (\mu_1, \mu_2)$. $\square$

6 Different support

So far, we have assumed that all bidders share the same support. Relaxing this assumption is difficult if there are several bidders, because it then becomes possible that not all bidders will have the same highest possible bid. However, with exactly two bidders the assumption of a common support is less important because both bidders must share a common maximal bid. Thus, we assume there are two bidders, and that bidder i's valuation is in the support $[\underline{v}_i, \overline{v}_i]$, $i = 1, 2$.

We assume that $\overline{v}_1 > \overline{v}_2$ and that there is some overlap of the supports, $\underline{v}_1 < \overline{v}_2$. As before, define $R_{1,2}(v) = F_2(v)/F_1(v)$ on the shared support of F_1 and F_2, $(\underline{v}, \overline{v}_2]$, where $\underline{v} = \max\{\underline{v}_1, \underline{v}_2\}$.

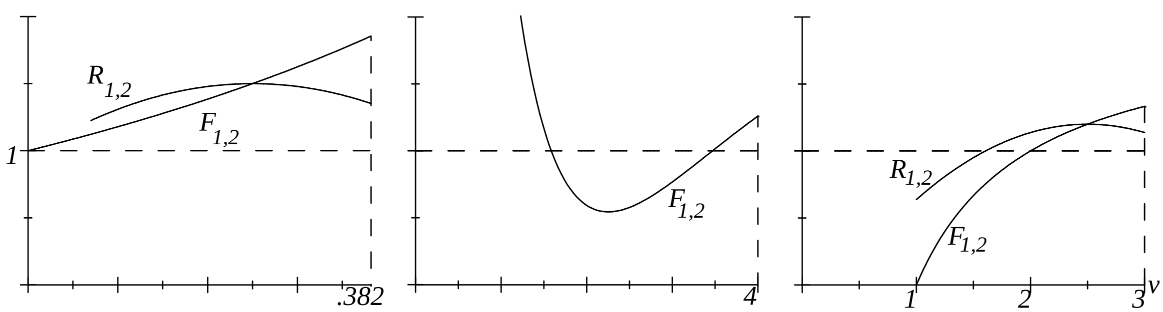
Maskin and Riley (2000a) and Hopkins (2007) show that when a bidder's valuation falls in the interval $(\underline{v}, \overline{v}_2]$ his bid is strictly increasing in his valuation and that he wins with strictly positive probability.[30] Hence, Lemma 3 is valid on $(\underline{v}, \overline{v}_2]$. We let $\underline{b}$ denote the lower end-point of the support of the winning bid. That is, the winning bid falls in the interval $[\underline{b}, \overline{b}]$.

Notice that $EU_2(\overline{v}_2) = \overline{v}_2 - \overline{b}$. However, bidder 1 with valuation $\overline{v}_2$ is better off, since he could earn the same payoff by imitating bidder 2. After all, a bid of $\overline{b}$ wins with probability one. Hence, $R_{1,2}(\overline{v}_2) > 1$. Further, since bidder 1 with valuation $\overline{v}_2$ bids less than bidder 2 with valuation $\overline{v}_2$, i.e. bids below $\overline{b}$, it must be the case that $R_{1,2}(\overline{v}_2) < F_{1,2}(\overline{v}_2)$. Notice that $F_{1,2}(\overline{v}_2) > 1$ since $F_2(\overline{v}_2) = 1 > F_1(\overline{v}_2)$.

In conclusion $R_{1,2}(\overline{v}_2)$ lies between 1 and $F_{1,2}(\overline{v}_2)$, and Lemma 3 applies to valuations in the shared support, $(\underline{v}, \overline{v}_2]$. Hence, the arguments in Section 3 can be used on valuations in $(\underline{v}, \overline{v}_2]$.

For example, Figure 4 highlights that Corollary 3 remains true. That is, if $F_{1,2}(v) = 1$ for some $v \in (\underline{v}, \overline{v}_2]$ then $R_{1,2}$ and $F_{1,2}$ must cross, implying bidding strategies must cross. In Figure 4, as we move leftward from $\overline{v}_2$, the decreasing function $R_{1,2}$ must intersect $F_{1,2}$ to the right of x_c.

In the following we consider the two cases that have received the most attention in the literature, $\underline{v}_1 = \underline{v}_2$ and $\underline{v}_1 < \underline{v}_2$, respectively.



(a) Maskin & Riley (2000a) (b) Weibull distributions (c) Kaplan & Zamir (2007)

Figure 4: Three examples with different supports.

[30]If $\underline{v}_1 \neq \underline{v}_2$ and v is sufficiently below $\underline{v}$, the bidder bids below $\underline{b}$ and never wins.

6.1 Same lower end-point; $\underline{v}_1 = \underline{v}_2$

Plum (1992) and Cheng (2006) consider situations where bidders draw valuations from different power distributions with a common lower bound. Given their assumptions, bidding strategies do not cross. Maskin and Riley (2000a, footnote 14), on the other hand, consider an example where

$$F_1(v) = 3v - 2v^2,\ v \in [0, 0.5]\,, \quad F_2(v) = 3v - v^2,\ v \in [0, 0.5(3 - \sqrt{5})].$$

In this case, $\overline{v}_1 > \overline{v}_2 \approx .382$ and $\underline{v}_1 = \underline{v}_2 = 0$. Notice that $F_{1,2}(0) = 1$. Even though F_1 first order stochastically dominates F_2, it is nevertheless the case that F_2 dominates F_1 in terms of the reverse hazard rate on the shared support, $[0, \overline{v}_2]$. In other words, $F_{1,2}$ is above one, and increasing, on $(0, \overline{v}_2]$. See Figure 4 (a).

Although $F_{1,2}$ is monotonic on $(0, \overline{v}_2]$ it is not the case that one bidder bids consistently more aggressively than another on $(0, \overline{v}_2]$. The reason is the "gap" between $F_{1,2}(\overline{v}_2)$ and 1 – the distribution of the bidder who is ostensibly "weaker" in the reverse hazard rate sense, bidder 1, has an extended support. Since $1 < R_{1,2}(\overline{v}_2) < F_{1,2}(\overline{v}_2)$, it must be the case that $R_{1,2}$ intersects $F_{1,2}$ exactly once as we move leftward. Hence, bidder 1 bids more aggressively for small valuations, while bidder 2 bids more aggressively for large valuations. In fact, whenever $f_1(0) = f_2(0) \in (0, \infty)$, bidding strategies must coincide somewhere on the interior, in line with the discussion following Corollary 3.

Notice that if F_1 is truncated on the smaller support $[0, \overline{v}_2]$ then bidder 1 would in fact be weaker everywhere and $F_{1,2}$ would be strictly increasing (the truncation leads $F_{1,2}(0)$ to fall below 1). Hence, bidder 1 would be consistently more aggressive than bidder 2. However, when the upper end-point of the support is extended, the number of times the bidding strategies crosses increases by one, from zero to one.

The same remark applies to Proposition 5. Consider, for example, the slight variation of Example 1, where

$$F_1(v) = \frac{1 - e^{-\left(\frac{v}{1.11}\right)^{1.5}}}{1 - e^{-\left(\frac{\overline{v}_1}{1.11}\right)^{1.5}}}, \quad F_2(v) = \frac{1 - e^{-\left(\frac{v}{1.5}\right)^{0.5}}}{1 - e^{-\left(\frac{4}{1.5}\right)^{0.5}}},$$

and $\underline{v}_1 = \underline{v}_2 = 0$, $\overline{v}_1 > \overline{v}_2 = 4$. Compared to Example 1, one bidder's support is extended to the right (the identities of bidders have been reversed compared to Example 1, because the change in support changes who is stronger near the top). As $\overline{v}_1$ increases, $F_{1,2}$ shifts up on the shared support, $[0, 4]$. Figure 4 (b) shows the final result. Given the analysis in Section 3 and the fact that $1 < R_{1,2}(\overline{v}_2) < F_{1,2}(\overline{v}_2)$, it is clear that $R_{1,2}$ will intersect $F_{1,2}$ twice on $[0, 4]$. Hence, bidding strategies cross *twice*. Proposition 5 and Theorem 2 relies on a common support; extending the upper end-point of one bidder's support adds the possibility of exactly one additional crossing.

6.2 Denser support; $\overline{v}_1 > \overline{v}_2 > \underline{v}_2 > \underline{v}_1$

Kaplan and Zamir (2007) analytically solve for bidding strategies when bidders draw valuations from different uniform distributions. One of their key examples is the following,

$$F_1(v) = \frac{v}{4},\ v \in [0,4], \quad F_2(v) = \frac{v-1}{2},\ v \in [1,3].$$

In this case bidder 1 is stronger near the top, $\overline{v}_1 > \overline{v}_2$, but the distribution functions cross at $v = 2$. See Figure 4 (c). Notice that $F_{1,2}$ is strictly increasing on $[1,3]$. The previous analysis clearly proves that bidding strategies must cross exactly once on this interval. Notice that this example is somewhat related to the analysis in Section 5. In particular, bidder 1 in Kaplan and Zamir's (2007) example is in some sense more unpredictable than bidder 2.

In fact, Kaplan and Zamir (2007) imposes a minimum bid (reserve price) of two, and find that bidding strategies cross once. Again, this is consistent with Figure 4 (c). The minimum bid effectively excludes bidders with valuation below 2, but since $F_{1,2}(2) = 1$ the result follows.[31]

Hopkins (2007) also consider cases in which one bidder's support is strictly within the support of another bidder. His focus is on bid distributions rather than the bidding strategies. By imposing conditions on the "dispersion" of the two distributions, he shows that *bid distributions cross exactly once*. In fact, his motivating example is the uniform distribution centered around the same mean, as in Kaplan and Zamir (2007).

This result nicely complements our Proposition 6, which also deals with cases where one distribution is in some sense more dispersed than another. For completeness, we prove a version of Hopkins' (2007) result here. First, we add the comment that given that $\overline{v}_1 > \overline{v}_2 > \underline{v}_2 > \underline{v}_1$ the bid distributions *must cross* at least once. Second, we give *different conditions* under which Hopkins' (2007) result that they cross *exactly once* holds true. Notice the similarity to Proposition 6.

Proposition 7 *Assume $\overline{v}_1 > \overline{v}_2 > \underline{v}_2 > \underline{v}_1$. Then, bid distributions cross at least once on $(\underline{b}, \overline{b})$ and p_1 has more mass at the tails than p_2. Moreover, if $m \leq 1$ on $(\underline{v}_2, \overline{v}_2]$ then $p_2(b)/p_1(b)$ is single peaked and crosses 1 exactly once.*

Proof. As Maskin and Riley (2000a) and Hopkins (2007) point out, bidder 2 with valuation $\underline{v}_2$ bids $\underline{b}$, implying that $p_2(\underline{b}) = 0 < p_1(\underline{b})$, where the inequality follows from the fact that bidder 1 submits bids with no chance of winning if his type

[31] Lemma 3 applies to the interval $(2,3]$ since bids are strictly increasing in valuation here (there is no mass of types bidding the minimum bid).

is sufficiently low. Moreover, as in the proof of Proposition 6, the derivative of $p_2(b)/p_1(b)$ is proportional to $\left(b_1^{-1}(b) - b\right)^{-1} - (b_2^{-1}(b) - b)^{-1}$. At bids above $b_1(\overline{v}_2)$, b_1^{-1} is larger than $\overline{v}_2$ which in turn is larger than b_2^{-1}, implying the derivative is negative for $b \geq b_1(\overline{v}_2)$. Thus, $p_2(b) > p_1(b)$ when b is large. Hence, p_1 and p_2 must cross and p_1 must have more mass at the tails than p_2. If $m \leq 1$ bidding strategies cross exactly once on $(\underline{v}_2, \overline{v}_2]$ (remember that $F_{1,2}(\overline{v}_2) > 1$ and $F_{1,2}(\underline{v}_2) = 0$, so any peak occurs where $F_{1,2} > 1$). Hence, the derivative changes sign exactly once between $\underline{b}$ and $b_1(\overline{v}_2)$. In other words, $p_2(b)/p_1(b)$ is single-peaked, meaning $p_2(b)/p_1(b) = 1$ exactly once. ■

7 Conclusion

In this paper we offered a new approach to the analysis of asymmetric first price auctions. Rather than looking at the system of differential equations determining bidding strategies, we started by comparing bidders' payoffs. This, in turn, allowed us to compare the distribution of bids submitted by various bidders, as well at the actual bidding strategies themselves.

The asymmetry between the bidders was permitted to take more general forms than those usually considered in the theoretical and numerical literature. Consequently, the most important existing results followed as corollaries of the two main results. In fact, we supplied new and simpler proofs of most existing theoretical results, and argued that the numerical research has been limited to a specific class of asymmetry.

More generally, we showed that the properties of the ratio of the distribution functions that describe beliefs can be used to derive upper bounds on the number of times bidders' expected payoffs and bidding strategies intersect. We described a class of situations in which the number of times bidding strategies cross can be precisely determined.

Given the focus on first order stochastic dominance in the current literature, we argued that the natural next step would be to more precisely characterize behavior in situations described by second order stochastic dominance. In the two-bidder case we showed that when second order, but not first order, stochastic dominance applies, payoffs and bidding strategies must necessarily cross, as must the distribution of bids.

We would also argue that an additional advantage of the approach suggested here is that it can be presented in easily digestible figures, implying that most of the arguments are visually obvious.

References

[1] Bajari, P., 2001, Comparing competition and collusion: A numerical approach, *Economic Theory* 18, 187–205.

[2] Blume, A. and Heidhues, P., 2004, All equilibria of the Vickrey auction, *Journal of Economic Theory*, 114, 170-177.

[3] Cantillon, E., 2004, The Effect of Bidders' Asymmetries on Expected Revenue in Auctions, forthcoming, *Games and Economic Behavior*.

[4] Cheng, H., 2006, Ranking sealed high-bid and open asymmetric auctions, *Journal of Mathematical Economics*, 42, Pages 471-498.

[5] Cheng, H., and Tan, G., 2007, Asymmetric Common Value Auctions with Applications to Auctions with Resale, mimeo.

[6] Fibich, G., and Gavious, A., 2003, Asymmetric First-Price Auctions - A Perturbation Approach, *Mathematics of Operations Research*, 28, 836-852.

[7] Fibich, G., Gavious, A., and Sela, A., 2002, Low and high types in asymmetric first-price auctions, *Economics Letters*, 75, 283-287.

[8] Gayle, W., and Richard, J-F, 2005, Numerical Solutions of Asymmetric First Price Independent Private Values Auctions, mimeo.

[9] Greismer, J., Levitan, R., and Shubik, M., 1967, Towards a Study of Bidding Processes, Part IV - Games with Unknown Costs, *Naval Research Logistics Quarterly*, 14, 415-434.

[10] Hopkins, E., 2007, Rank-Based Methods for the Analysis of Auctions, mimeo.

[11] Hopkins, E. and Kornienko, T., 2007, Cross and Double Cross: Comparative Statics in First Price and All Pay Auctions, *The B.E. Journal of Theoretical Economics*: Vol. 7: Iss. 1 (Topics), Article 19.

[12] Kaplan, T. R., and Zamir, S., 2007, Asymmetric First-Price Auctions with Uniform Distributions: Analytic Solutions to the General Case, mimeo.

[13] Kirkegaard, R., 2006, A Simple Approach to Analyzing Asymmetric First Price Auctions, mimeo.

[14] Krishna, V., 2002, Auction Theory, *Academic Press*.

[15] Lebrun, B., 1998, Comparative Statics in First Price Auctions, *Games and Economic Behavior*, 25, 97-110.

[16] Lebrun, B., 1999, First Price Auctions in the Asymmetric N Bidder Case, *International Economic Review*, 40, 125-142.

[17] Lebrun, B., 2006, Uniqueness of the equilibrium in first-price auctions, *Games and Economic Behavior*, 55, 131–151.

[18] Li, H., and Riley, J., 2006, Auction Choice, forthcoming, *International Journal of Industrial Organization.*

[19] Marshall, R.C., Meurer, M.J., Richard, J-F, and Stromquist, W., 1994, Numerical Analysis of Asymmetric First Price Auctions, *Games and Economic Behavior*, 7, 193-220.

[20] Maskin, E. and Riley, J., 2000a, Asymmetric Auctions, *Review of Economic Studies*, 67, 413-438.

[21] Maskin, E. and Riley, J., 2000b, Equilibrium in Sealed High Bid Auctions, *Review of Economic Studies*, 67, 439-454.

[22] Maskin, E. and Riley, J., 2003, Uniqueness of equilibrium in sealed high-bid auctions, *Games and Economic Behavior*, 45, 395-409.

[23] Milgrom, P., 2004, Putting Auction Theory to Work, *Cambridge University Press.*

[24] Myerson, R.B., 1981, Optimal Auction Design, *Mathematics of Operations Research*, 6, 58-73.

[25] Plum, M., 1992, Characterization and Computation of Nash-Equilibria for Auctions with Incomplete Information, *International Journal of Game Theory*, 20, 393-418.

[26] Riley, J.G. and Samuelson, W.F., 1981, Optimal Auctions, *American Economic Review*, 71, 381-392.

[27] Vickrey, W., 1961, Counterspeculation, Auctions, and Competitive Sealed Tenders, *Journal of Finance*, 16, 8-37.